Engineering Science Series

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PROBLEMS IN MECHANICS

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Based on the original collection of I. V. Mestchersky

BY

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FOREWORD

A marked increase of interest in mechanics as applied to engineering is noticeable in technical circles as well as in the engineering colleges. A number of schools have introduced recently more extensive courses in theoretical mechanics. There is a distinct trend toward a more mature and comprehensive presentation of this important subject. The course in mechanics usually offers the student the first opportunity to apply his basic training in mathematics and physics to problems of a practical character.

The exposition of the principles and theorems of mechanics is of little practical value to the student unless he is constantly exercised in their application to actual problems. Only by this means can mechanics become a working tool for the future engineer.

This volume contains a collection of problems covering statics, kinematics and dynamics, arranged in a systematic way. The problems are preceded by a brief concise outline of the theorems which are used in their solution. The outline is not intended to take the place of an extended exposition of the subject, but is merely offered for convenient reference.

The first part of the collection covers problems in plane and space statics. The section on plane statics includes a number of problems on trusses and cables; problems on friction are segregated in a separate group, since this subject seems to present special difficulty to students. Problems on the first and second moments of areas are included in the section on centers of gravity.

The second part of the collection covers the kinematics of a point and the kinematics of a rigid body, in rotation about a fixed axis and motion parallel to a fixed plane. These are followed by problems in relative motion of a point and in composition of rotations of a rigid body.

The first sections of the third part of the problems cover the application of the differential equations of Newton to the motion of particles and to rotation and plane motion of rigid bodies. The following sections contain problems involving the application

of the principles of work and energy, impulse and momentum, and motion of the center of gravity. Special sections on bearing reactions, vibration and oscillation, and impact are included. A table of units and trigonometric functions is added for convenience.

This group of problems grew out of the collection published by the late I. V. Mestchersky, of the Polytechnic Institute of St. Petersburg. In assembling the original collection, Mestchersky had the collaboration of his many assistants, among whom were engineers of prominence in various fields. Two of these collaborators, Professors S. Timoshenko and B. A. Bakhmeteff are well known to American engineers. It was Mestchersky's desire for many years to see his problems translated into English. While the present collection is based on Mestchersky's work, it differs from it in several respects. The original problems were reworded to suit American practice, which involved changing the units and numerical values from the metric system. They were issued by Columbia University in this form. In the present volume the problems were rearranged and their number was increased by 40 per cent. The added problems are of intermediate difficulty, a type which was not well represented in the original collection. (Several of these problems were taken from the files of Professor C. H. Burnside, of Columbia University.) The theorems of mechanics did not exist in the original.

The problems are of a wide range of difficulty. A certain number of typical problems, about 10 per cent of the total, are furnished with solutions. This was done to suggest to students a method of attack which they can follow to advantage in handling the rest of the problems. Answers to nearly all problems are given. Much care was taken in checking the correctness of the solutions and answers. However, the authors realize that errors will exist. They will appreciate any assistance which readers may render in pointing out detected errors.

For the convenience of instruction, many problems in statics as well as in dynamics, include the suggestion that the several methods available for solution should be applied.

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April 1939

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RESUMÉ OF MECHANICS

PART I. STATICS

FUNDAMENTAL PRINCIPLES

1. A force has magnitude, direction, and a point of application. The force can be represented by a vector which indicates the direction, and whose length may represent the magnitude of the force to a chosen scale.

2. Two forces applied to a rigid body (or a particle) are in equilibrium when they are equal in magnitude, opposite in direction and act along the same line.

3. The application of any system of forces in equilibrium to a rigid body does not in any way affect the state of rest or motion of that body.

3a. In a rigid body the point of application of a force may be shifted along the line of action without changing the effect of the force.

4. Whenever a body exerts a force on a second body, the second body exerts an equal and oppositely directed force on the first body.

4a. Where a body rests on supports, the supports may be replaced by reaction forces acting on the body at the supporting points.

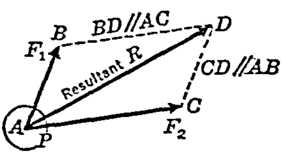


FIG. 1

5. Two non-collinear forces acting on a particle are equivalent to a single resultant force. The magnitude and direction of this single force are represented by the diagonal of a parallelogram constructed on the two forces (Fig. 1).

PLANE STATICS

Composition of Concurrent Forces.

6. The resultant of several collinear forces acts along the same line and is equal to the algebraic sum of the forces. (The forces acting in one sense are taken positive and those in the opposite sense negative.) The sense of the resultant is indicated by the algebraic sign of the sum.

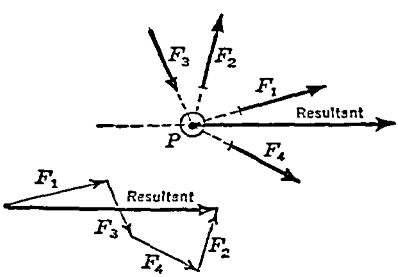


FIG. 2

7. The resultant of several concurrent forces is the vector sum of all the forces. The vector drawn from the origin to the tip of the last arrow in the force polygon (Fig. 2) represents the resultant. The resultant passes through the point of concurrency of the forces.

16 A system of parallel forces is in equilibrium when the algebraic sum of the forces and the algebraic sum of the moments of the forces about any point in the plane of the forces are both equal to zero The conditions of equilibrium are

$$\Sigma F = 0, \qquad \Sigma M = 0 \quad \text{(about any point)}$$

Graphical Composition of Non-concurrent Coplanar Forces.

17. When the "space diagram" for the forces (a scale drawing showing the lines of action of the forces in their true relative positions) is given, Fig 9(a) and Fig 10(a), the line of action of the resultant can be found by a graphical construction Construct the force polygon

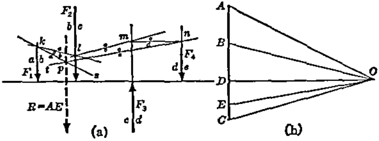


Fig 9

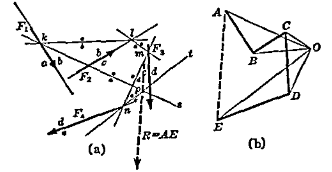


Fig 10

$ABCDE$ The magnitude and direction of the resultant force are represented by the vector AE Choose arbitrarily a pole O and draw rays from O to A, B, C, D, and E, Figs 9(b) and 10(b) Through any point k on the force F_1 (ab), draw a line kl parallel to the ray OB to an intersection with the force F_2 (bc) at point l Similarly draw lines lm and mn, parallel to the rays OC and OD From the points k and n draw lines ks and nt parallel to the rays OA and OE The resultant R passes through the point of intersection p of these two lines

17a. If the force polygon closes, the resultant force is zero The final lines ks and nt are then parallel to each other, and E coincides with

A. The resultant couple is equal to a force represented by the length of ray AO (or OE) times the perpendicular distance between lines ks and nt.

18. It is necessary and sufficient for the equilibrium of a coplanar force system that the force polygon, Figs. 9(b) and 10(b), closes and that the final lines ks and nt in the space diagram, Figs. 9(a) and 10(a), coincide.

Algebraic Composition of General Coplanar Forces.

19. In any coplanar system of forces the component of the resultant parallel to any axis NN is equal to the algebraic sum of the n-components of all the given forces. Using two rectangular coordinate axes to determine the magnitude and direction of the resultant, we see that the x and y components of R will be (Fig. 11):

$$R_x = \Sigma F_x, \qquad R_y = \Sigma F_y, \qquad R = \sqrt{R_x^2 + R_y^2},$$

$$\sin\theta = \frac{R_y}{R}, \qquad \text{or} \qquad \tan\theta = \frac{R_y}{R_x}.$$

The line of action of the resultant is determined by the principle of moments. The moment of the resultant R about any point A in the plane of the forces is equal to the algebraic sum of the moments of the given forces about A. The moment of R about A is ΣM_a.

19a. If the reference point A is chosen at the origin O and if for the forces $F_1, F_2, \cdots, F_n$, respective points (x_1, y_1), (x_2, y_2) $\cdots$ (x_n, y_n) in their lines of action are known, the moment M_0 is:

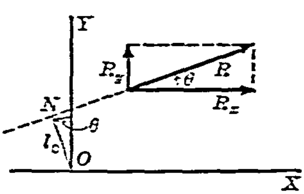


FIG. 11

$$M_0 = (F_{1y}\cdot x_1 - F_{1x}\cdot y_1) + (F_{2y}\cdot x_2 - F_{2x}\cdot y_2) + \cdots + (F_{ny}\cdot x_n - F_{nx}\cdot y_n), \quad l_0 = \frac{M_0}{R}.$$

20. Any system of coplanar forces can be replaced by the single force R passing through any point A in the plane and a couple whose moment is equal to ΣM_a. The force R is the same for all points in the plane but ΣM_a depends on the location of the point A.

20a. If the force R is equal to zero, the resultant is a couple of moment ΣM_a, which in this case is independent of the location of the point A.

21. It is necessary and sufficient for the equilibrium of a coplanar force system that the resultant force be equal to zero, and the resultant moment about any point in the plane of the forces be equal to zero:

$$\Sigma F_x = 0; \qquad \Sigma F_y = 0; \qquad \Sigma M_a = 0.$$

STATICS IN SPACE

Concurrent Forces in Space

22 The magnitude and direction of the resultant of several concurrent forces in space are given by the closing side of a space polygon of forces The line of action of the resultant passes through the point of intersection of the component forces

The component of the resultant force R_n parallel to any axis NN is equal to the algebraic sum of the components of the given forces parallel to NN

$$R_n = F_{1n} + F_{2n} + F_{3n} + F_{4n} + \cdots = \Sigma F_n$$

23 The resultant of several concurrent forces is usually obtained by taking any three perpendicular axes OX, OY, OZ, and finding the components of the resultant parallel to these axes The x, y, z components of each given force F (Fig 12) are given by

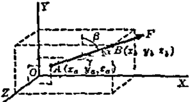


FIG 12

$$F_x = F \cos \alpha, \qquad F_y = F \cos \beta,$$
$$F_z = F \cos \gamma,$$

where α, β, γ are the angles between the force F and the x, y, and z axes, respectively The cosines of these angles are called the direction cosines They are related by the equation $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$ If points A (x_a, y_a, z_a) and B (x_b, y_b, z_b) are two points on the line of action of the force F, the direction cosines of the line AB are

$$\cos \alpha = \frac{x_b - x_a}{L}, \qquad \cos \beta = \frac{y_b - y_a}{L}, \qquad \cos \gamma = \frac{z_b - z_a}{L},$$

where

$$L = \sqrt{(x_b - x_a)^2 + (y_b - y_a)^2 + (z_b - z_a)^2}$$

The components of the resultant are

$$R_x = \Sigma F_x, \qquad R_y = \Sigma F_y, \qquad R_z = \Sigma F_z$$

The magnitude of the resultant is

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2}$$

Its direction is determined by

$$\cos \alpha_R = \frac{R_x}{R}, \qquad \cos \beta_R = \frac{R_y}{R}, \qquad \cos \gamma_R = \frac{R_z}{R}.$$

23a. Concurrent forces in space are in equilibrium when and only when

$$\Sigma F_x = 0, \qquad \Sigma F_y = 0, \qquad \Sigma F_z = 0,$$

that is, when the resultant is zero

Couples in Space.

24. Any couple acting on a rigid body can be replaced by another couple acting in a plane parallel to the plane of the given couple, provided the moments of both couples are equal in magnitude and have the same sense.

A couple may be represented by a vector normal to the plane of the couple. The magnitude of the couple is represented by the length of the vector to an arbitrary scale. The direction of the vector is such that the moment is clockwise looking in the direction in which the vector points.

24a. A system of several couples is equivalent to a resultant couple, the vector of which is the vector sum of the component couples considered as vectors.

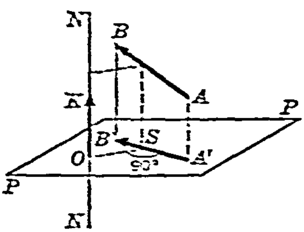


FIG. 13

25. The moment of a force AB with respect to an axis NN (Fig. 13) is equal to the product of the projection $A'B'$ of the force on a plane P normal to the axis NN, and the perpendicular distance OS between the axis and the line of action of the projected force $A'B'$. OS is equal to the shortest distance between NN and AB. The vector OK of the moment is parallel to the axis NN.

25a. If a force F passes through a point x, y, z and has components F_x, F_y, and F_z, its moments about the coordinate axes OX, OY, and OZ are:

$$M_x = y\cdot F_z - z\cdot F_y; \qquad M_y = z\cdot F_x - x\cdot F_z; \qquad M_z = x\cdot F_y - y\cdot F_x.$$

26. A force F in space acting through a point K can be resolved into an equal force acting through any specified point A (Fig. 14) and a couple lying in the plane SS containing the force and the point A. The magnitude of the couple is equal to the product of the force and the perpendicular distance h from the point A to the line of action of the force. Taking three rectangular coordinate axes X, Y, and Z through the point A, the force at A has the same axial components as the original force:

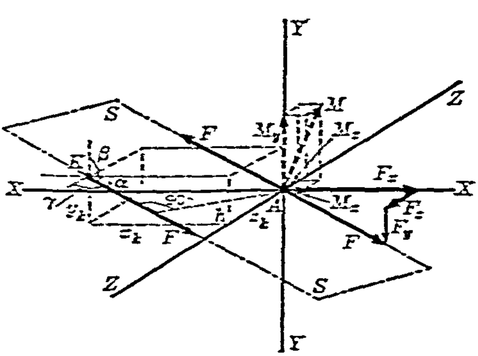


FIG. 14

$$F_x = F\cos\alpha, \qquad F_y = F\cos\beta, \qquad F_z = F\cos\gamma.$$

The moment vector M of the couple has axial components

$$M_x = y\ F_z - z\ F_y, \qquad M_y = z\ F_x - x\ F_z, \qquad M_z = x\ F_y - y\ F_x,$$

where x, y, z, are the coordinates of point K

General Case of a System of Forces in Space.

27. *Any system of forces can be reduced to a resultant force acting through a specified point and a resultant couple* Taking three rectangular coordinate axes X, Y, and Z through the specified point, the components of the resultant force will be

$$R_x = F_{1x} + F_{2x} + F_{3x} + \qquad = \Sigma F_x,$$
$$R_y = F_{1y} + F_{2y} + \qquad = \Sigma F_y,$$
$$R_z = F_{1z} + F_{2z} + \qquad = \Sigma F_z,$$
$$R = \sqrt{R_x^2 + R_y^2 + R_z^2},$$
$$\cos \alpha_R = \frac{R_x}{R}, \qquad \cos \beta_R = \frac{R_y}{R}, \qquad \cos \gamma_R = \frac{R_z}{R}$$

The axial components of the vector representing the resultant couple are given by the formulas

$$C_x = M_{1x} + M_{2x} + \qquad = \Sigma M_x,$$
$$C_y = M_{1y} + M_{2y} + \qquad = \Sigma M_y,$$
$$C_z = M_{1z} + M_{2z} + \qquad = \Sigma M_z$$

The magnitude of the resultant moment is

$$C = \sqrt{C_x^2 + C_y^2 + C_z^2}$$

The resultant couple C lies in a plane which is perpendicular to the moment vector whose direction cosines are

$$\cos \alpha_M = \frac{C_x}{C}, \qquad \cos \beta_M = \frac{C_y}{C}, \qquad \cos \gamma_M = \frac{C_z}{C}$$

27a The resultant force R is the same for all reference points A in space, but the resultant couple depends on the location of the point A

If the resultant force R is zero, that is, if $R_x = R_y = R_z = 0$, the resultant couple has the same value for any reference point in space

28 It is necessary and sufficient for the equilibrium of any force system that the resultant force and the resultant moment both be equal to zero The conditions of equilibrium are expressed by the six equations

$$R_x = \Sigma F_x = 0, \qquad R_y = \Sigma F_y = 0 \qquad R_z = \Sigma F_z = 0,$$
$$C_x = \Sigma M_x = 0, \qquad C_y = \Sigma M_y = 0, \qquad C_z = \Sigma M_z = 0$$

28a. In analyzing a system of forces in equilibrium, use is made of the fact that in such a system the algebraic sum of the components of all the given forces parallel to any axis is zero, and the algebraic sum of the moments of all the given forces about any axis is zero. The proper choice of reference axes simplifies considerably the necessary computations.

Simplest Equivalent Forms of Force Systems.

29. A system of forces is equivalent to a couple when the resultant force is zero, that is, when $R_x = R_y = R_z = 0$.

30. A system of forces is equivalent to one resultant force when the resultant couple C is either zero ($C_x = C_y = C_z = 0$), or acts in a plane parallel to the resultant force R (the vector C is normal to the force R). In the latter case, the equivalent force is found as shown in § 14. When the force R and the vector C are perpendicular, the following relation exists between their direction cosines:

$$\cos \alpha_R \cdot \cos \alpha_M + \cos \beta_R \cdot \cos \beta_M + \cos \gamma_R \cdot \cos \gamma_M = 0.$$

31. When both the resultant force and resultant couple are not zero, the system of forces can be reduced to two forces. This can be accomplished by resolving the resultant couple into two equivalent forces, and combining one of these with the force R.

31a. The system can always be reduced to a force and a couple acting in a plane normal to the force. The system is then said to be reduced to the "canonical form."

The method of reducing a system to the canonical form is as follows: Find the resultant force R and the resultant couple C for an arbitrarily chosen system of coordinate axes. The force R is represented by the vector OA and the couple C by the vector OD (Fig. 15). Resolve the couple C into components $OF = C_2$ acting along OA, and $OE = C_1$ perpendicular to OA. Replace the couple C_1, which is equal to $C \sin (\angle C, R)$, by two forces OB and $O'G$, making both equal to R, with the perpendicular distance between their lines of action equal to $OO' = C_1/R$. OO' is perpendicular to the plane AOD. OA and OB balance each other, and the system is reduced to a force $O'G = R$ and a couple $OF = C_2$ lying in a plane perpendicular to the force $O'G$. The line of action of $O'G$ is called the central axis of the system. The

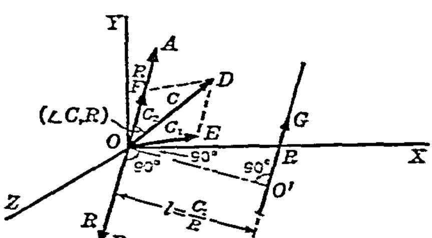


FIG. 15

couple C_2 of the system of forces for this axis is the smallest resultant couple possible

CENTER OF GRAVITY. FIRST AND SECOND MOMENTS

Center of Gravity.

32. *The resultant of the distributed gravity forces acting on all* particles of the body, irrespective of the orientation of the body, passes *through a point called the center of gravity of the body* For every position of the body, the algebraic sum of the moments of the distributed *gravity forces with respect to any axis passing through the center of* gravity is equal to zero

33 If a body can be divided into several parts, such that for each of these parts the weight w_i and the coordinates x_i, y_i, z_i of the center of gravity are known (Fig 16), then the coordinates $\bar{x}$, $\bar{y}$, $\bar{z}$ of the center of gravity of the entire body are given by the following equations

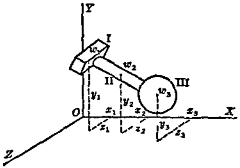


Fig 16

$$\bar{x} = \frac{\Sigma x_i w_i}{W},$$

$$\bar{y} = \frac{\Sigma y_i w_i}{W},$$

$$\bar{z} = \frac{\Sigma z_i w_i}{W},$$

where $W = \Sigma w_i$ is the weight of the entire body

34 When the specific weight (q) at any point can be expressed as a function of the coordinates, the center of gravity can be found by the following equations

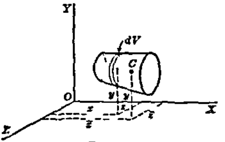


Fig 17

$$\bar{x} = \frac{\int_V x(q)dV}{W},$$

$$\bar{y} = \frac{\int_V y(q)dV}{W},$$

$$\bar{z} = \frac{\int_V z(q)dV}{W},$$

where V is the volume of the body (Fig 17)

35. When the density is uniform, the above equations become:

$$\bar{x} = \frac{\int_V x\,dV}{V}, \qquad \bar{y} = \frac{\int_V y\,dV}{V}, \qquad \bar{z} = \frac{\int_V z\,dV}{V}.$$

Centroid.

36. The point given by the equations of § 35 is the centroid of the volume of the body. For a body of uniform density the center of gravity coincides with its centroid.

37. The coordinates of the centroid of a plane area with respect to axes lying in the plane of the area are given by the following equations:

$$\bar{x} = \frac{\int_A x\,dA}{A}, \qquad \bar{y} = \frac{\int_A y\,dA}{A}.$$

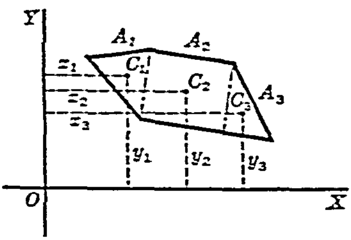


FIG. 18

37a. When the total area can be divided into parts (Fig. 18), such that for each part the area A_i and the location of the centroid are known, then the above equations become:

$$\bar{x} = \frac{\Sigma A_i \cdot x_i}{A}, \qquad \bar{y} = \frac{\Sigma A_i \cdot y_i}{A}.$$

38. When a body has a point, a line, or a plane of symmetry, the centroid of the body is at the point, on the line, or in the plane of symmetry. This is also true for a plane area.

First Moment of Area.

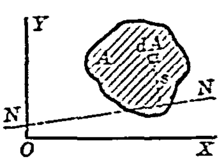


FIG. 19

39. The first moment of an area A with respect to any axis NN is the integral $\int_A s \cdot dA$, where s is the perpendicular distance from the axis NN to the element of area dA (Fig. 19). With respect to the coordinate axes X and Y the first moments are, respectively,

$$Q_x = \int_A y \cdot dA = A \cdot \bar{y}, \qquad Q_y = \int_A x \cdot dA = A \cdot \bar{x}.$$

Second Moment. Moment of Inertia.

40. The second moment of the area A (Fig. 19) with respect to axis NN is the integral $\int_A s^2 \cdot dA$. This second moment is commonly called

the moment of inertia of the area With respect to the coordinate axes X and Y the moments of inertia are, respectively,

$$I_x = \int_A y^2 \, dA, \qquad I_y = \int_A x^2 \, dA.$$

40a The moment of inertia of an area with respect to any axis is equal to the sum of the moments of inertia of its parts with respect to the same axis

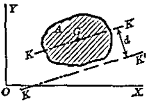


Fig 20

41. The moment of inertia of an area (Fig 20) with respect to any axis $K'K'$ in the plane of the area, is equal to the moment of inertia of the area with respect to an axis KK parallel to $K'K'$ and passing through the centroid of A, plus the area of A times the square of the perpendicular distance d between the axes

$$I_{k'} = I_k + Ad^2$$

Product of Inertia.

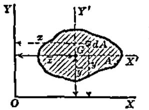


Fig 21

42 The product of inertia of an area A with respect to coordinate axes X and Y which lie in the plane of the area, is given by the integral,

$$\int_A x \; y \; dA = P_{xy} \qquad \text{(Fig 21)}$$

43. *The product of inertia of area A with respect to coordinate axes X and Y is equal to* $P_{xy} = \bar{P}_{x'y'} + A\bar{x}\bar{y}$, where $\bar{P}_{x'y'}$ is the product of inertia of area A with respect to axes X' and Y', passing through the centroid of the area and parallel to axes X and Y, and where $\bar{x}$ and $\bar{y}$ are the coordinates of the centroid of area A

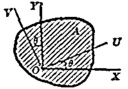


Fig. 22

44. The moments of inertia of an area A with respect to axes U and V inclined at an angle θ to axes X and Y (Fig 22), are, respectively

$$I_u = I_x \cos^2\theta + I_y \sin^2\theta - 2P_{xy} \sin\theta\cos\theta,$$
$$I_v = I_x \sin^2\theta + I_y \cos^2\theta + 2P_{xy} \sin\theta\cos\theta$$

The product of inertia of area A with respect to axes U and V is

$$P_{uv} = \tfrac{1}{2}(I_x - I_y)\sin 2\theta + P_{xy}\cos 2\theta$$

PART II. KINEMATICS

MOTION OF A POINT

Path.

46 *The line traced by a point in motion is called the path of the point* When the path is given, the motion is completely defined if the distance of the moving point measured along the path from a fixed point on the path is known for every instant t The relationship between s and t may be expressed analytically $s = f(t)$, or graphically

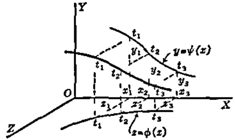


FIG 23

47 If the positions of a moving point are expressed by three coordinates x, y, and z as functions of the time t, the path is completely defined by the three equations $x = f_1(t)$, $y = f_2(t)$, $z = f_3(t)$ (Fig 23) Eliminating t from these equations, the projections of the path on two coordinate planes are obtained, for example $y = \psi(x)$, $z = \phi(x)$ These two equations define the path as shown in Fig 23

48 *When the point moves in a plane two coordinates determine the motion*, for example $x = f_1(t)$ and $y = f_2(t)$ Eliminating t from these *expressions, the equation of the path* $y = \phi(x)$ *is obtained*

Velocity of a Point.

49. The displacement of a point during a time interval $\Delta t = t_2 - t_1$, is the vector distance $P_1P_2 = \Delta s$ between the positions of the point at the beginning and end of the time interval (Fig 24)

FIG 24

50 The average velocity for a time interval Δt is the *ratio of the displacement* to the time interval, $\Delta s/\Delta t$ The velocity at any instant is $v = ds/dt$ *The velocity for any position* of the point is directed along the tangent to the path at that point and is in the *direction of motion The velocity is a vector quantity* Its magnitude is called the speed of the point Velocities can be added and subtracted *by adding and subtracting their vectors*

51. If the motion of a point is defined by the relations $x = f_1(t)$, $y = f_2(t)$, $z = f_3(t)$, the projections of the velocity of the point on the

coordinate axes are

$$v_x = \frac{dx}{dt} = f_1'(t), \qquad v_y = \frac{dy}{dt} = f_2'(t), \qquad v_z = \frac{dz}{dt} = f_3'(t).$$

The magnitude of the velocity is $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$. The direction of the velocity (and of the tangent to the path) is given by the direction cosines:

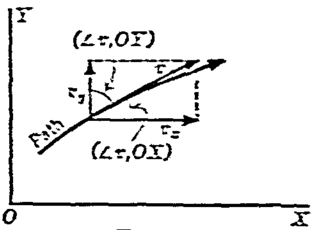


FIG. 25

$$\cos(\angle v, OX) = v_x/v,$$
$$\cos(\angle v, OY) = v_y/v,$$
$$\cos(\angle v, OZ) = v_z/v.$$

51a. In plane motion: $v_x = dx/dt = f_1'(t)$, $v_y = dy/dt = f_2'(t)$, $v = \sqrt{v_x^2 + v_y^2}$, and $\cos(\angle v, OX) = v_x/v$, $\cos(\angle v, OY) = v_y/v$ (Fig. 25).

Acceleration of a Point.

52. The average acceleration during a time interval Δt is the ratio $\Delta v/\Delta t$, where Δv is the change in velocity for the interval (Fig. 26).

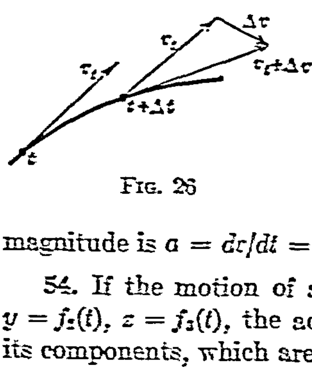


FIG. 26

The acceleration at any instant is the time rate of change of the velocity (the limit of $\Delta v/\Delta t$ as Δt approaches zero). Accelerations can be added or subtracted by adding or subtracting their vectors.

53. In rectilinear motion the acceleration is directed along the path of motion; its magnitude is $a = dv/dt = d^2s/dt^2 = f''(t)$, where $s = f(t)$.

54. If the motion of a point is defined by the relations $x = f_1(t)$, $y = f_2(t)$, $z = f_3(t)$, the acceleration a of the point can be found from its components, which are

$$a_x = \frac{d^2x}{dt^2} = f_1''(t), \qquad a_y = \frac{d^2y}{dt^2} = f_2''(t), \qquad a_z = \frac{d^2z}{dt^2} = f_3''(t),$$

and its magnitude is $a = \sqrt{a_x^2 + a_y^2 + a_z^2}$. The direction of the acceleration is determined from the direction cosines

$$\cos(\angle a, OX) = \frac{a_x}{a}, \qquad \cos(\angle a, OY) = \frac{a_y}{a}, \qquad \cos(\angle a, OZ) = \frac{a_z}{a}.$$

54a. For plane motion the equations are $x = f_1(t)$, $y = f_2(t)$, $a_x = d^2x/dt^2 = f_1''(t)$, $a_y = d^2y/dt^2 = f_2''(t)$, $a = \sqrt{a_x^2 + a_y^2}$, and $\cos(\angle a, OX) = a_x/a$.

Normal and Tangential Accelerations

55 For curvilinear motion of a point on a plane when the path is known and the motion is defined by $s = f(t)$, the acceleration may be determined by two components the tangential component $a_t = dv/dt = d^2s/dt^2$, and the normal component $a_n = v^2/\rho$, where ρ is the radius of curvature of the path at the point

55a In the case of a point moving with a constant speed v on the circumference of a circle of radius R, the acceleration is v^2/R and is directed toward the center of the circle (centripetal acceleration)

55b The total acceleration of a point M (Fig 27) moving along any path ss lies always in the plane PP passing through the velocity vector of the point, i e, through the line TT tangent to the path at M, *and through the center of* curvature Q of the path (The osculating plane of the path at M) The radius of curvature $QM = \rho$ is normal to the tangent TT The total acceleration $a = \sqrt{a_x^2 + a_y^2 + a_z^2}$ can be resolved into two components one, a_t, directed along the tangent TT and called the tangential acceleration, the other, a_n, directed toward the center of curvature Q and called the normal acceleration The tangential acceleration gives the rate of increase of the magnitude of the velocity of the point M and is equal to $a_t = dv/dt$ The normal acceleration is equal to $a_n = v^2/\rho$, where v is the velocity of point M, and ρ is the radius of curvature of the path at M The total acceleration is

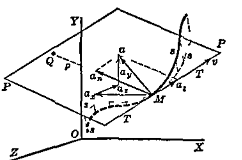


Fig 27

$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{\left(\frac{dv}{dt}\right)^2 + \frac{v^4}{\rho^2}}$$

Motion in Polar Coordinates

56 The motion of a point in a plane can be defined in polar coordinates by the relations $r = f_1(t)$ and $\theta = f_2(t)$ The radial component of the velocity along the radius vector is $v_r = dr/dt = f_1'(t)$, the transverse component of velocity, normal to the radius vector, is

$$v_\theta = r\,d\theta/dt = rf_2'(t), \qquad \text{and} \qquad v = \sqrt{v_r^2 + v_\theta^2}$$

radial and transverse components of the acceleration a_r and a_θ and the absolute value of the acceleration a, are

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2, \qquad a_\theta = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right), \qquad a = \sqrt{a_r^2 + a_\theta^2}$$

56a. The relations between polar and orthogonal coordinates, when the two systems have the same origin and the angle θ is measured from the x axis, are as follows: $x = r\cos\theta$, $y = r\sin\theta$. When $r = f_1(t)$ and $\theta = f_2(t)$, the equation of the path in terms of x and y is obtained by eliminating t from the above equations.

Integrals of Motion.

57. If the components of the acceleration of a point are known as functions of time, the velocity and path of the point can be found by integration. The formulas are

$$a_x = \frac{d^2x}{dt^2} = f_1(t), \qquad v_x = \int a_x dt = \int f_1(t)dt + C_1 = F_1(t) + C_1,$$

$$x = \int v_x dt = \int F_1(t)dt + C_1 t + C_2.$$

The constants of integration are evaluated from two known conditions at given times, such as two positions, or a position and a velocity. The components of the acceleration parallel to the other coordinate axes are treated in a similar way.

57a. If the components of the velocity of a point are given as functions of time, the path of the point can be found by integration. It is necessary to know the position at some instant of time to evaluate the single constant of integration.

57b. For rectilinear motion with a constant acceleration a, the velocity is $v = v_0 + at$; and the distance from some reference point is $s = s_0 + v_0 t + \frac{1}{2}at^2$, where v_0 and s_0 are the velocity and the distance at time $t = 0$. When $v_0 = s_0 = 0$, $s = \frac{1}{2}at^2$, and $v = at$.

Velocity Hodograph.

58. The line HH described by the end of the radius vector OH, which represents at any instant the magnitude and direction of the velocity v of a point M moving along a path ss (Fig. 28), is the velocity-hodograph of the moving point. If the motion of the point M is defined by its coordinates $x = f_1(t)$, $y = f_2(t)$, $z = f_3(t)$, the position of the end H of the radius vector OH at any instant is determined by the equations $x_1 = f_1'(t)$, $y_1 = f_2'(t)$, $z_1 = f_3'(t)$. Elimination of t from these equations gives the equation of the hodograph HH.

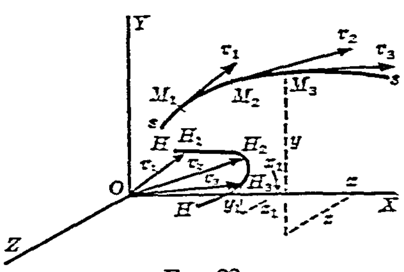


FIG. 28

59 The velocity of the point H describing the velocity hodograph of a moving point M is equal, at any instant, to the total acceleration of the point M, in magnitude and in direction

MOTION OF A RIGID BODY

Translation.

60 *A body has a motion of translation when the paths of all its points are parallel* Any straight line in the body remains parallel to its original position throughout the motion The velocities as well as the accelerations are the same for all the points, at any instant The motion of the body is completely defined by the motion of any one of its points

Rotation about a Fixed Axis

61 When a body rotates about a fixed axis, the motion is defined by the angle of rotation $\theta = f(t)$ between two planes, both passing through the axis, one attached to the body and the other fixed in space

62 The time rate of change of the angle of rotation is the angular velocity of the body $\omega = d\theta/dt = f'(t)$

63 The time rate of change of the angular velocity ω is the angular acceleration of the body $\alpha = d\omega/dt = d^2\theta/dt^2 = f''(t)$

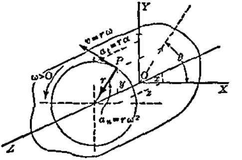


Fig 29

64 The path of every point P (Fig 29) in a rigid body rotating about a fixed axis is a circle lying on a plane which is perpendicular to the axis, and having its center on the axis

64a The velocity of a point P (Fig 29) at any instant is directed along the tangent to the circular path of the point, and has the value $v = r\omega$

64b The acceleration of a point P (Fig 29) of the rotating body at any instant consists of two components the tangential component $a_t = r\alpha$ directed along the tangent, in a sense to agree with the sense of α, and the normal component $a_n = r\omega^2 = v^2/r$, directed toward the axis of rotation

65 When the motion of a point P is referred to a fixed orthogonal system of coordinates, with the axis of rotation as the OZ axis and with the fixed reference plane as ZOX, the coordinates and the components

of velocity and acceleration for point P (Fig. 29) will be as follows:

$x = r\cos\theta$; $y = r\sin\theta$; z.

	x-components	y-components
Velocity $v = r\omega$	$-r\omega\sin\theta = -y\omega$	$r\omega\cos\theta = x\omega$
Tangential Comp. Accel $a_t = r\alpha$	$-y\alpha$	$x\alpha$
Normal Comp. Accel. $a_n = r\omega^2 = \frac{v^2}{r}$	$-x\omega^2$	$-y\omega^2$
Total Accel. $a = \sqrt{a_t^2 + a_n^2} = r\sqrt{\alpha^2 + \omega^4}$	$-y\alpha - x\omega^2$	$x\alpha - y\omega^2$

Motion of a Rigid Body Parallel to a Fixed Plane.

66. The path of motion of any point B (Fig. 30) lies in the plane of cross section KK which passes through B and is parallel to the fixed plane. The motion of point B is defined by the motion of any base point A in section KK and the rotation of B about A. The velocity of point B is the vector sum of the velocity v_A of point A and the velocity $v_{B/A} = q\omega$ of B with respect to A, where q is the fixed length between A and B and ω is the angular velocity of the body. The acceleration of point B is the vector sum of the acceleration a_A of point A and the acceleration $a_{B/A}$ of B with respect to A; the acceleration $a_{B/A}$ is the vector sum of its tangential component $(a_{B/A})_t = q\alpha$ and normal component $(a_{B/A})_n = q\omega^2$, where α is the angular acceleration of the body.

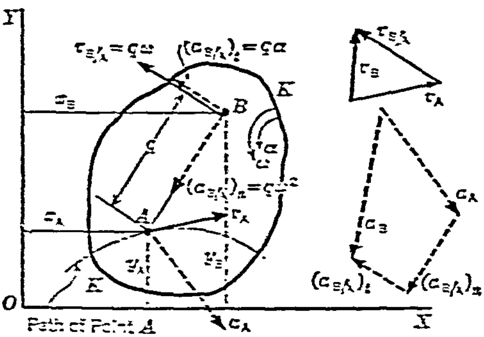


FIG. 30

66a. The letter ω denotes the instantaneous angular velocity, and α the instantaneous angular acceleration of any line in the section KK.

67. If the velocities of two points, A and B (Fig. 30), in the cross-section KK are known, the angular velocity of the body can be found by dividing the relative velocity $v_{B/A}$ by the distance between the two points, $\omega = [(v_{B/A})/q]$.

68. When a fixed xy coordinate system is taken in the plane of the section KK (Fig. 30) or in any parallel plane, and the motion of the base point A is given by the two equations, $x_A = f_1(t)$ and $y_A = f_2(t)$, the motion of point B is determined by the formulas

$$x_B = x_A + q\cos\theta, \qquad y_B = y_A + q\sin\theta,$$

where θ is the angle between the x axis and the line AB, expressed as a function of time $\theta = f_3(t)$

The x and y components of the velocity of the point B are

$$v_{Bx} = v_{Ax} - q\omega \sin\theta, \qquad v_{By} = v_{Ay} + q\omega \cos\theta,$$

where $v_{Ax} = f_1'(t)$, $v_{Ay} = f_2'(t)$, and $\omega = d\theta/dt = f_3'(t)$ The x and y components of the acceleration of the point B are

$$a_{Bx} = a_{Ax} - q\omega^2 \cos\theta - q\alpha \sin\theta, \quad \text{and} \quad a_{By} = a_{Ay} - q\omega^2 \sin\theta + q\alpha\cos\theta,$$

where $a_{Ax} = f_1''(t)$, $a_{Ay} = f_2''(t)$, and $\alpha = d^2\theta/dt^2 = f_3''(t)$

Instantaneous Center

69 Any change in position of a plane figure, in its own plane, may be accomplished by a rotation of the plane figure about a center located somewhere on the plane

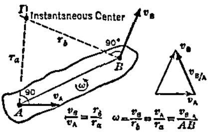


Fig 31

Every infinitesimal displacement of the plane figure during its motion can be accomplished by a rotation about an instantaneous center If the directions of the velocities of two points in the plane figure are known, the instantaneous center is found as the intersection of lines drawn through the points normal to the velocities (*Fig 31*) *The velocity of any point* in the plane figure is equal to the product of the radius from the instantaneous center to the point and the angular velocity of the plane figure It is directed normal to the radius

70 The locus of the instantaneous centers in space is called the space centrode The locus of the instantaneous centers on the extended plane which moves with the figure is called the body centrode At any moment the two centrodes are tangent to each other, at the instantaneous center for the moment, and the motion of the plane figure can be reproduced by rolling the body centrode on the space centrode without slipping

70a A body moving parallel to a fixed plane has an instantaneous axis of rotation, perpendicular to the fixed plane and passing through the instantaneous center of any cross-section of the body taken parallel to the fixed plane The instantaneous axes generate two cylindrical surfaces called the space axode and body axode, corresponding to space and body centrodes for the cross-section

RELATIVE MOTION OF A POINT

71. When the motion of a point M is known with respect to a body BB, which itself moves with respect to an arbitrary system of fixed coordinate axes XYZ (Fig. 32), the absolute motion s'-s' of the point, i.e., its motion with respect to the fixed coordinate axes, is determined as the resultant of the relative motion of the point M with respect to the reference body BB, and of the motion of transportation of M, i.e., the absolute motion of that point of the body BB at which M is located at the instant.

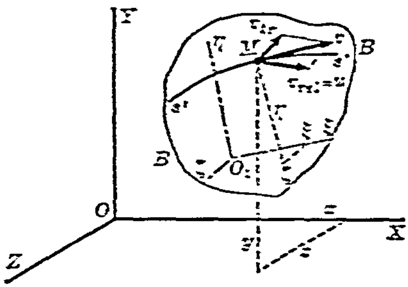


FIG. 32

The relative motion is conveniently defined by relative coordinates $\xi = \psi_1(t)$, $\eta = \psi_2(t)$, $\zeta = \psi_3(t)$, where the coordinate axes $O_1\xi$, $O_1\eta$, $O_1\zeta$ are rigidly attached to, and move with, the reference body BB. The path of the motion of M with respect to the ξ, η, ζ axes, is found by elimination of t from the coordinate equations. The relative velocity u of point M is determined by its components: $u_\xi = d\xi/dt$; $u_\eta = d\eta/dt$; $u_\zeta = d\zeta/dt$, and $u = \sqrt{u_\xi^2 + u_\eta^2 + u_\zeta^2}$. The relative acceleration b of the point M is determined by its components: $b_\xi = d^2\xi/dt^2$; $b_\eta = d^2\eta/dt^2$; $b_\zeta = d^2\zeta/dt^2$, and $b = \sqrt{b_\xi^2 + b_\eta^2 + b_\zeta^2}$.

Coriolis Acceleration.

72. The absolute velocity v of the point M is the vector sum of its relative velocity v_{rel} and of the velocity v_{tr} of that point of the reference body at which M is located at the instant. The velocity v_{tr} is called the velocity of transportation of M.

The total absolute acceleration a of the point M is the vector sum of its relative acceleration a_{rel}, of the acceleration a_{tr} of that point of the reference body at which M is located at the instant (which is called the acceleration of transportation of M), and of an additional component a_{cor}, the Coriolis acceleration. This additional component, the Coriolis acceleration, exists only when the reference body BB has a motion of rotation about some axis, and it vanishes when the motion of BB is a translation. The magnitude of the Coriolis acceleration a_{cor} is given by the formula $a_{cor} = 2 \times u' \times \omega$, where ω is the angular velocity, u' is the projection of the relative velocity v_{rel} on a plane CC normal to the instantaneous axis of rotation NN of body BB. The direction of the Coriolis acceleration is normal to the plane ML containing v_{rel} and u', while its sense is determined by the direction

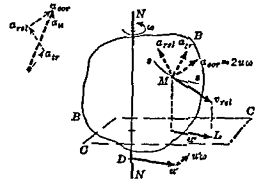


FIG 33

of rotation of the point L of the vector u'. It is convenient to find the Coriolis acceleration by imagining the vector u' attached to the axis of rotation NN at an arbitrary point D (Fig 33) and letting it rotate with the body BB at an angular velocity ω Twice the velocity of the vector point L gives the Coriolis acceleration of the point M, in magnitude and direction

72a If the relative velocity v_{rel} lies in a plane normal to the axis NN, then $a_{cor} = 2 \times v_{rel} \times \omega$ If the relative velocity v_{rel} is parallel to the axis NN, then $a_{cor} = 0$

Projections of Velocity and of Acceleration.

73 The projection v_l of a velocity v on an arbitrary axis l-l (Fig 34), making angles α, β, γ with the coordinate axes OX, OY, OZ, respectively, is

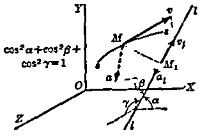


FIG. 34

$$v_l = v \cos(\angle v, l)$$
$$= v_x \cos\alpha + v_y \cos\beta + v_z \cos\gamma$$

The projection a_l of the acceleration a of the point M on the axis l-l is

$$a_l = a \cos(\angle a, l)$$
$$= a_x \cos\alpha + a_y \cos\beta + a_z \cos\gamma$$

If the direction of line l-l is variable, two relations $\alpha = \phi_1(t)$ and $\beta = \phi_2(t)$ are sufficient to define its position at any instant, since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

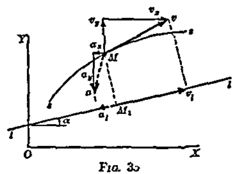


FIG. 35

74 When the point M moves in a plane and coordinate axes are taken in that plane, the projection v_l of the velocity v on an arbitrary axis l-l in the plane of motion is (Fig 35) $v_l = v_x \cos\alpha + v_y \sin\alpha$, where α is the angle between the x axis and line l-l The projection a_l of the acceleration a is $a_l = a_x \cos\alpha + a_y \sin\alpha$.

COMPOSITION OF ROTATIONS OF A BODY

75. Rotation of a body may be represented by a vector whose length is equal to the instantaneous angular velocity of the body, drawn to an arbitrarily chosen scale. The line of action of the vector is parallel to the axis of rotation of the body. Its direction is such that the rotation is clockwise when viewed in the direction in which the vector points.

76. The motion of a body subjected to rotation around several axes simultaneously is equivalent to a resultant rotation which is the vector sum of the component rotations considered as vectors.

77. The angular velocity of a body rotating simultaneously in the same direction around two parallel axes is equal to the sum of the component angular velocities. The instantaneous axis of the resultant rotation is parallel to the axes of the component rotations, lies in their plane, and cuts any line intersecting them into parts inversely proportional to the angular velocities of the component rotations.

The angular velocity of a body rotating simultaneously around two parallel axes in opposite directions is equal to the difference of the component angular velocities. The instantaneous axis of rotation is parallel to the axes of the component rotations, lies in their plane, outside these axes, on the side of the one with higher angular velocity. It cuts any line intersecting the axes at a point whose distances from the axes are inversely proportional to the angular velocities of the component rotations.

78. Simultaneous rotation of a body around two parallel axes with the same angular velocity but in opposite directions results in a translatory motion of the body. It moves in a direction normal to the plane of the axes of the component rotations, with a velocity equal to the product of the distance between the axes times the angular velocity of the component rotations.

ROTATION OF A RIGID BODY AROUND A FIXED POINT

79. Any change in position of a body which has one point fixed may be accomplished by a rotation about an axis passing through the fixed point. A continuous motion may be reproduced by a series of infinitely small rotations about a series of instantaneous axes which all pass through the fixed point. The instantaneous axis can be found if the directions of the velocities of any two points in the body are known. The intersection of the planes passing through the points, perpendicular to the directions of the velocities, is the instantaneous axis. The locus of the instantaneous axes in space forms a fixed conical surface called

the space axode The locus of the instantaneous axes in the body forms a moving conical surface called the body axode The motion of the body can be reproduced by rolling the body axode over the space axode

80. The motion of a rigid body rotating around a fixed point O (Fig 36) is fully defined at any instant by the direction of its instantaneous axis of rotation NN and by its instantaneous angular velocity It is convenient to choose a system of fixed coordinate axes OX, OY, OZ, with the origin in the fixed point The rotation of the body around NN is equivalent to a simultaneous rotation around the fixed coordinate axes OX, OY, OZ, with instantaneous component angular velocities ω_x, ω_y, ω_z, respectively The instantaneous angular velocity ω is equal to $\omega = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$ The direction cosines of the instantaneous axis of rotation NN are $\cos\alpha = \omega_x/\omega$, $\cos\beta = \omega_y/\omega$, $\cos\gamma = \omega_z/\omega$ The instantaneous axis of the body is defined by the equation $x_1/\omega_x = y_1/\omega_y = z_1/\omega_z$, where x_1, y_1, z_1 are the coordinates of a point on the axis The velocity of a point A with coordinates x, y, z is determined by the equations

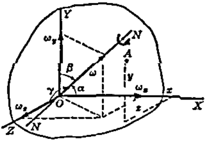


FIG 36

$$v_x = z\omega_y - y\omega_z, \qquad v_y = x\omega_z - z\omega_x, \qquad v_z = y\omega_x - x\omega_y,$$
$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}.$$

PART III. KINETICS

FUNDAMENTAL PRINCIPLES

81. A particle either remains at rest or continues to move along a straight line with constant velocity, unless it is acted upon by an external force.

The time rate of change of velocity, i.e., the acceleration of the particle, is proportional to the force causing it and has the same direction as the force.

82. The coefficient of proportionality between a force F and the acceleration a which it imparts to a particle is the mass of the particle. With a proper choice of units, $F = ma$, or $a = F/m$, or $m = F/a$.

82a. The acceleration of a particle caused by several simultaneous forces is the vector sum of the accelerations imparted by each force.

82b. If F_x, F_y, F_z are components of the force in a system of orthogonal coordinate axes, the components of the acceleration of the particle are determined by the formulas

$$ma_x = m\frac{d^2x}{dt^2} = F_x, \qquad ma_y = m\frac{d^2y}{dt^2} = F_y, \qquad ma_z = m\frac{d^2z}{dt^2} = F_z.$$

When several forces are acting simultaneously on a particle the components of the acceleration are determined by the equations

$$ma_x = \Sigma F_x, \qquad ma_y = \Sigma F_y, \qquad ma_z = \Sigma F_z.$$

Units.

83. In engineering, the English-speaking countries commonly use the foot-pound-second system or the inch-pound-second system of units. In scientific work, the so-called absolute or centimeter-gram-second (C.G.S.) system of units is used. (Absolute systems use mass as a basic concept in contradistinction to engineering systems, which use force as a basic concept.)

In engineering, the unit of force is one pound. (A force of one pound is the weight of, or the earth's gravitational pull on, the "standard pound body" of the Bureau of Standards when measured at 45° latitude and at sea level, in vacuum.)

To impart an acceleration a to a body weighing w pounds, a force $F = (w/g)a$ pounds is necessary, where both a and g are taken either in in./sec.^{2} or in ft./sec.^{2}, and g is the acceleration caused by the force of gravity measured at 45° latitude and at sea level. For engineering

purposes, g is taken to be 386 in./sec.^{2} or 32.2 ft./sec.^{2} The factor w/g is the mass m of the body, its units are lbs. sec.^{2}/in or lbs. sec^2/ft

In the absolute C G.S system of units, the unit of mass is one gram One gram is approximately the mass of 1 cubic centimeter of water at 4° centigrade The force necessary to impart an acceleration of 1 cm /sec^2 to a mass of one gram is one dyne, which is the absolute unit of force. The acceleration of gravity $g = 981$ cm /sec^2 at 45° latitude and at sea level, therefore the weight of one gram mass is 981 dynes. This weight is also called one gram To impart an acceleration of a cm./sec^2 to a body weighing m grams requires a force of $F = ma$ dynes $= ma/g$ grams.

RECTILINEAR MOTION OF A PARTICLE

Equation of Motion.

84 A particle moves in a straight line only when the resultant of all forces acting on it is directed along the line of motion If the line of motion be taken as a coordinate axis OX, then the equation of motion may be expressed as $F_x = ma_x = (w/g)d^2x/dt^2$

The force may be constant or variable.

Integration of the Equation of Motion.

85 If the force is constant, $F_x = F$, then we have

$$\frac{w}{g}\frac{d^2x}{dt^2} = F, \qquad \frac{d^2x}{dt^2} = \frac{g}{w}F,$$

where w is the weight of the particle and g is the acceleration of gravity The velocity and position of the particle as a function of time are determined by successive integrations of the equation of motion

$$v = \frac{dx}{dt} = \frac{g}{w}Ft + C,$$

$$x = \frac{1}{2}\frac{g}{w}Ft^2 + Ct + D$$

The integration constants C and D are evaluated from known conditions of motion at one or two arbitrary instants of time If the distance x_0 and the velocity v_0 are known at the instant $t = 0$ the integration constants are $D = x_0$ and $C = v_0$. Then

$$v = v_0 + \frac{g}{w}Ft, \qquad x = x_0 + v_0t + \frac{1}{2}\frac{g}{w}Ft^2$$

85a. When the force is expressed as a function of the time, $F_x = f(t)$, the motion is defined by the equation $(w/g)d^2x/dt^2 = m\, d^2x/dt^2 = f(t)$,

or $d^2x/dt^2 = (1/m)f(t)$. Then

$$v = \frac{dx}{dt} = \left(\frac{1}{m}\right)\int f(t)dt + C = \left(\frac{1}{m}\right)F(t) + C,$$

where $F(t) = \int f(t)dt$. Moreover, $x = (1/m)\int F(t)dt + Ct + D$. The constants of integration C and D are evaluated as above.

85b. When the force is expressed as a function of position, $F_x = f(x)$, the motion is defined by the equation $(w/g)d^2x/dt^2 = m\cdot d^2x/dt^2 = f(x)$, or $d^2x/dt^2 = (1/m)f(x)$. Since

$$\frac{d}{dt}(v^2) = 2v\frac{dv}{dt} = 2v\frac{d^2x}{dt^2},$$

the equation of motion can be written in the form

$$\frac{d}{dt}(v^2) = 2v\frac{1}{m}f(x) = \frac{2}{m}\frac{dx}{dt}f(x), \qquad \text{or} \qquad d(v^2) = \frac{2}{m}f(x)dx.$$

Integrating this, we find $v^2 = (2/m)\int f(x)dx + C = (2/m)F(x) + C$, and $v = dx/dt = \pm\sqrt{(2/m)F(x) + C}$, where the sign is chosen to satisfy the initial conditions. A second integration gives

$$t = \pm\int\frac{dx}{\sqrt{(2/m)F(x) + C}} + D.$$

This equation gives the relation between x and t. The constants of integration C and D are evaluated as before.

85c. When the force is expressed as a function of the velocity $F_x = f(v)$, the motion is defined by the equation $(w/g)d^2x/dt^2 = m\cdot d^2x/dt^2 = f(v)$. Therefore we have $d^2x/dt^2 = dv/dt = (1/m)f(v)$. Integrating, we find $t = m\int dv/f(v) + C = mF(v) + C$. Solving algebraically for v, we find $v = \phi(t)$, then $dx = \phi(t)dt$, and $x = \int\phi(t)dt + D$. The constants of integration C and D are evaluated as before.

CURVILINEAR MOTION OF A PARTICLE

86. The motion in space of a particle of weight w is specified by the coordinates x, y, z of the particle with respect to three arbitrarily chosen fixed coordinate axes. If a force F, with components F_x, F_y, and F_z,

is acting on the particle, the equations of motion are

$$\frac{w}{g}\frac{d^2x}{dt^2} = m\frac{d^2x}{dt^2} = F_x, \qquad \frac{w}{g}\frac{d^2y}{dt^2} = m\frac{d^2y}{dt^2} = F_y, \qquad \frac{w}{g}\frac{d^2z}{dt^2} = m\frac{d^2z}{dt^2} = F_z$$

The force F may be constant or variable Each of these three equations is integrated as indicated in §§ 85a, 85b, and 85c, and three equations of motion are obtained $x = F_1(t)$, $y = F_2(t)$, $z = F_3(t)$

Plane Motion

87 A particle moves on a plane only when the resultant of all forces acting on the particle lies in the plane of motion If the motion of the particle is referred to two coordinate axes x and y in the plane, the equations of motion are

$$\frac{w}{g}\frac{d^2x}{dt^2} = m\frac{d^2x}{dt^2} = F_x, \qquad \frac{w}{g}\frac{d^2y}{dt^2} = m\frac{d^2y}{dt^2} = F_y$$

By integration the coordinates are obtained in the form

$$x = F_1(t), \qquad y = F_2(t)$$

The equation of the path can be found by eliminating t from these two equations

87a The motion of a particle under the action of gravity alone is confined to a vertical plane which includes the initial velocity vector Taking one coordinate axis OX horizontal, and the second OY vertical, with the positive direction upward, the equations of motion give $d^2x/dt^2 = 0$, $d^2y/dt^2 = -g$ The path is a parabola with its axis vertical

Motion under a Central Force

88 The motion of a particle acted upon solely by a central force (either of attraction or repulsion), i e, a force whose direction always passes through a fixed point, is confined to a plane This plane passes through the initial velocity vector and through *the center of the force* The components of the central force F (Fig 37) are

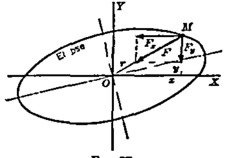


Fig 37

$$F_x = F\frac{x}{r},$$

and

$$F_y = F\frac{y}{r},$$

where x and y are the coordinates of the particle, and $r = \sqrt{x^2 + y^2}$

When the central force is an attraction proportional to the distance from the center O, $F = kr$, the equations of motion are

$$\frac{w}{g}\frac{d^2x}{dt^2} = -kr\left(\frac{x}{r}\right) = -kx, \qquad \frac{w}{g}\frac{d^2y}{dt^2} = -kr\left(\frac{y}{r}\right) = -ky,$$

or

$$\frac{d^2x}{dt^2} + \frac{(gk)}{w}x = 0, \qquad \frac{d^2y}{dt^2} + \frac{(gk)}{w}y = 0.$$

The motion consists of two superimposed harmonic oscillations at right angles to each other (§ 136); the path is an ellipse.

KINETICS OF A SYSTEM OF PARTICLES

System of Particles.

89. A system of particles is a number of particles considered together. The particles of the system may be free or geometrically interrelated to each other. A system in which the distance between every pair of particles remains constant is a rigid body.

Each particle of a system is generally under the action of forces which may be divided into impressed or external forces acting from without, and into internal forces resulting from mutual actions of the particles of the system upon each other.

Effective and Inertia Force.

90. The effective force for a particle is the vector quantity whose magnitude is $\epsilon = ma$, and which has the same direction as the acceleration a of the particle. The quantity ma is measured in units of force. At each instant of motion the effective force is equal to and collinear with the resultant of all actual forces (external and internal) applied to the particle.

90a. In a moving system of particles, the resultant of the effective forces for all the particles of the system is identical with the resultant of all the external forces applied to the system.

91. The inertia force for a particle is the vector quantity whose magnitude is $i = ma$ and which acts opposite to the direction of the acceleration a of the particle. At each instant of motion the inertia force is equal to and directed opposite to the resultant of all actual forces applied to the particle.

91a. In a moving system of particles, the inertia forces for all the particles are at any instant in equilibrium with all the external forces applied to the system (Principle of D'Alembert).

Rigid Body. Motion of Translation

92 In the case of translation of a rigid body of weight W, the resultant effective force is equal to $(W/g)a$ It passes through the center of gravity of the body and acts in the direction of the acceleration a The resultant R of all external forces acting on the body must therefore pass through the center of gravity of the body, act in the direction of the acceleration, and be equal to $R = (W/g)a$

92a The resultant inertia force in this case is equal to $(W/g)a$ passes through the center of gravity of the body, and acts in a direction opposite to the acceleration a The resultant R of all the external forces is in equilibrium with the resultant inertia force $R - (W/g)a = 0$

Moment of Inertia

93 The moment of inertia I_n of a rigid body about any axis NN is equal to the sum of the products of the masses dw/g of all particles of the body, each times the square of its distance r from the axis $I_n = \int_W (dw/g)r^2 = (1/g)\int_W r^2\,dw$ When the specific weight q is uniform throughout the body, $I_n = (q/g)\int_V r^2\,dv$

94 A length k_n, such that $(W/g)k_n^2 = I_n$, is called the radius of gyration of the body for axis NN, the entire weight W concentrated at a distance k_n from the axis would have the same moment of inertia as the body

Parallel Axis Theorem

95 The moment of inertia I_n of a body about any axis NN is equal to the moment of inertia $\bar{I}_n$ about an axis parallel to NN and passing through the center of gravity of the body, plus the product of the mass W/g of the body times the square of the perpendicular distance c between the two axes

$$I_n = \bar{I}_n + \frac{W}{g}\,c^2$$

96 If OX, OY, OZ are three coordinate axes in a body of specific weight q the moments of inertia about these axes are, respectively

$$I_x = \frac{q}{g}\int_V (y^2 + z^2)dv, \qquad I_y = \frac{q}{g}\int_V (z^2 + x^2)dv, \qquad I_z = \frac{q}{g}\int_V (x^2 + y^2)dv,$$

where x, y, z are the coordinates of the elemental volume dv of the body

Rotation about a Fixed Axis.

101. In a rigid body rotating about a fixed axis with an angular velocity ω and an angular acceleration α, the effective force for each particle lies in the plane of motion for the particle, has a normal component $e_n = mr\omega^2$, directed toward the axis of rotation, and a tangential component $e_t = mr\alpha$, in the sense determined by α

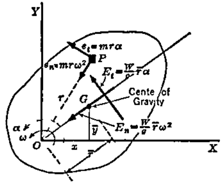


Fig 39

The resultant of all the normal effective force-components in the body is $E_n = (W/g)\bar{r}\omega^2$, and it lies in a plane that passes through the axis of rotation and the center of gravity of the body The resultant of all the tangential effective force-components in the body is given by $E_t = (W/g)\bar{r}\alpha$ and is perpendicular to the plane passing through the axis of rotation and the center of gravity of the body The algebraic sum of the moments of all effective forces in the body about the axis of rotation is $T_e = I_0\alpha$, where I_0 is the moment of inertia of the body about the axis of rotation

101a If the body has a plane of symmetry normal to the axis of rotation, the resultant effective force-components E_n and E_t lie in that plane If we take the axes OX and OY in the plane of symmetry, with the origin O at the axis of rotation, the axial components of the resultant effective force are

$$E_x = -\frac{W}{g}\bar{x}\omega^2 - \frac{W}{g}\bar{y}\alpha, \qquad E_y = -\frac{W}{g}\bar{y}\omega^2 + \frac{W}{g}\bar{x}\alpha, \qquad T_0 = T_e = I_0\alpha$$

102 The resultant of all the external forces F applied to the body (including the reactions) must be equal to, and collinear with, the resultant of the effective force system

$$\Sigma F_x = E_x, \qquad \Sigma F_y = E_y, \qquad M_0 = I_0\alpha,$$

where M_0 is the algebraic sum of the moments of the external forces applied to the body about the axis of rotation

102a When the axis of rotation passes through the center of gravity, $E_x = E_y = 0$, the resultant of the effective force system is a couple $T_e = \bar{I}_0\alpha$ In this case, $\Sigma F_x = 0$, $\Sigma F_y = 0$, $M_0 = \bar{I}_0\alpha$

103. The resultant S of the inertia forces for all the particles of the body is equal and opposite to the resultant of the effective forces

$$S_x = -E_x, \qquad S_y = -E_y, \qquad T_s = -T_e,$$

where T_s is the algebraic sum of the moments of all inertia forces in the body about the axis of rotation. Correspondingly, the resultants of the normal and tangential components of the inertia forces S_n and S_t are $S_n = -E_n$, $S_t = -E_t$. (The normal component S_n is sometimes called the centrifugal force on the body.)

104. The external forces (including the reactions) applied to the body are in equilibrium with the inertia forces. The equations can therefore be written in the form

$$\Sigma F_x + S_x = \Sigma F_x + \left(\frac{W}{g}\bar{x}\omega^2 + \frac{W}{g}\bar{y}\alpha \right) = 0,$$

$$\Sigma F_y + S_y = \Sigma F_y + \left(\frac{W}{g}\bar{y}\omega^2 - \frac{W}{g}\bar{x}\alpha \right) = 0,$$

$$M_C + T_s = M_C - I_C\alpha = 0.$$

Plane Motion of a Rigid Body.

105. In a rigid body having plane motion, the acceleration for any particle lies in its plane of motion. The acceleration of a particle P, Fig. 40(a), is the vector sum of the acceleration of an arbitrary base point A and the acceleration due to relative rotation of P with respect to A. (See § 66.)

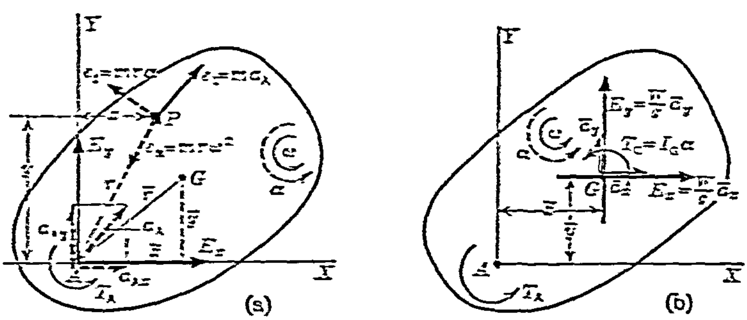


FIG. 40

Taking coordinate axes AX and AY parallel to the fixed plane, with the origin at the base point A, the axial components of the effective force for the particle P are

$$e_x = ma_{Ax} - m(x\omega^2 + y\alpha), \qquad e_y = ma_{Ay} - m(y\omega^2 - x\alpha),$$

where a_{Ax} and a_{Ay} are the axial components of the acceleration of point A.

106. If the body has a plane of symmetry parallel to the fixed plane, the resultant of the effective forces for all the particles lies in the plane

of symmetry The axial components of the resultant effective force E are

$$E_x = \frac{W}{g} a_{Ax} - \frac{W}{g}(\bar{x}\omega^2 + \bar{y}\alpha), \qquad E_y = \frac{W}{g} a_{Ay} - \frac{W}{g}(\bar{y}\omega^2 - x\alpha),$$

where W is the weight of the body The algebraic sum of the moments of all effective forces in the body about an axis passing through a point A normal to the plane of symmetry is

$$T_A = I_A\alpha + \bar{x}\frac{W}{g} a_{Ay} - \bar{y}\frac{W}{g} a_{Ax},$$

where I_A is the moment of inertia of the body about the axis through A

106a When the center of gravity G is selected as the reference base point, the expressions for the resultant effective force reduce to the simple form $E_x = (W/g)a_x$, $E_y = (W/g)a_y$, and the moment is $T_G = \bar{I}_G\alpha$

107. The resultant of all the external forces F applied to the body (including the reactions) must be equal to and collinear with the resultant of the effective force system

$$\Sigma F_x = E_x, \qquad \Sigma F_y = E_y, \qquad M_A = T_e,$$

where M_A is the algebraic sum of the moments of the external forces applied to the body about the axis through the point A

107a If the center of gravity G is selected as the reference point, we have $\Sigma F_x = (W/g)a_x$, $\Sigma F_y = (W/g)a_y$ and $M_G = I_G\alpha$

107b If the resultant effective force components E_x and E_y and the moment T_G are determined for the base point at the center of gravity, the moment T_A of the resultant effective force about any point A, see Fig 40(b), is given by the formula

$$T_A = \bar{I}_G\alpha + \frac{W}{g}\bar{a}_y\,\bar{x} - \frac{W}{g}\bar{a}_x\,y = M_A,$$

where M_A is the moment of the external forces about the point A

108 The resultant S of the inertia forces for all the particles of the body is equal and opposite to the resultant of the effective forces

$$S_x = -E_x, \qquad S_y = -E_y \qquad T_s = -T_e,$$

where T_s is the algebraic sum of the moments of all inertia forces in the body about the axis through the point A

109 The external forces (including the reactions) applied to the body are in equilibrium with the inertia forces When referred to a

base point A, the equations of motion are

$$\Sigma F_x + S_x = \Sigma F_x - \frac{W}{g} a_{Ax} + \frac{W}{g}(\bar{x}\omega^2 + \bar{y}\alpha) = 0,$$

$$\Sigma F_y + S_y = \Sigma F_y - \frac{W}{g} a_{Ay} + \frac{W}{g}(\bar{y}\omega^2 - \bar{x}\alpha) = 0,$$

$$M_A + T_z = M_A - I_A\alpha - \bar{x}\frac{W}{g} a_{Ay} + \bar{y}\frac{W}{g} a_{Ax} = 0.$$

109a. When the center of gravity is taken as a base point, the equations become

$$\Sigma F_x - \frac{W}{g}\bar{a}_x = 0, \qquad \Sigma F_y - \frac{W}{g}\bar{a}_y = 0, \qquad M_G - \bar{I}_G\alpha = 0.$$

WORK AND KINETIC ENERGY

Work.

110. When a force F acts on a particle which moves along any path (Fig. 41), the work dW done by the force during a differential displacement ds is given by $dW = F\cos\phi \cdot ds$, where ϕ is the angle between the force F and the tangent to the path at P. The total work which is done by the force during the motion of the particle from position A to position B is

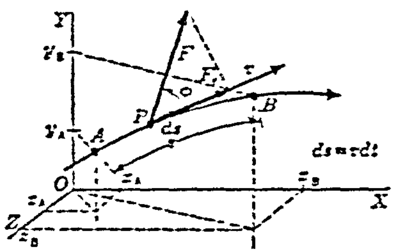


FIG. 41

$$W = \int_A^B F\cos\phi\, ds = \int_A^B F_t \cdot ds,$$

where F_t is the working component of the force.

The work is taken as positive when the working component is in the direction of motion ($\phi < 90°$), and negative when the working component is opposite to the direction of motion ($90° < \phi < 180°$).

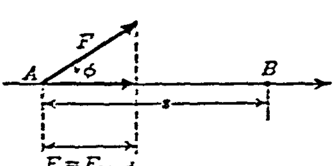


FIG. 42

110a. When a force F, constant in magnitude and direction, acts on a particle which moves along a straight line (Fig. 42), the product of the distance s between the initial and final positions of the particle and the working component of the force F is the work W done by the force: $W = sF_t = sF\cos\phi$. When $\phi = 0$, $W = s \times F$.

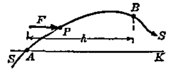


FIG 43

110b The work done by a force F which is constant in magnitude and direction, acting on a particle P (Fig 43) which moves along any path SS from point A to point B, is equal to the distance h between the projections of the points A and B on the line AK parallel to F, multiplied by the force $W = h \times F$

111 The work done by a force F with axial components F_x, F_y and F_z acting on a particle during a differential displacement ds which has projections dx, dy, and dz, is $dW = F_x dx + F_y dy + F_z dz$ The total work done during a finite displacement of the particle along its path from point A to point B is given by the formula

$$W = \int_{x_A}^{x_B} F_x \, dx + \int_{y_A}^{y_B} F_y \, dy + \int_{z_A}^{z_B} F_z \, dz$$

112 The work done by a system of forces acting simultaneously on a rigid body, or any other system of particles, is equal to the algebraic sum of the works done by the several forces during the displacement of their points of application

113 The work done by a constant couple M during an angular displacement of the couple in its plane is equal to the product of the couple and the angular displacement $W = M \theta$, where θ is the angular displacement measured in radians

Power

114 The time rate at which work is done is called power The power P at any instant can be determined by the equation $P = dW/dt$, where W is the work expressed as a function of time t If the work is done at a constant rate, the power $P = W/(t_2 - t_1)$, where W is the total work done during the time interval from t_1 to t_2

114a When a force F acts on a particle which moves with a velocity v, the power at any instant is $P = F \quad v \cos (\angle F, v)$

114b When a body rotates with an angular velocity ω, *and a mo*ment M acts upon the body in a plane normal to the axis of rotation, the power produced by M at any instant is $P = M\omega$

Kinetic Energy

115 The product of one-half the mass of a particle and the square of its velocity is the kinetic energy of the particle $E = \frac{1}{2}mv^2 = \frac{1}{2}(w/g)v^2$

116 The kinetic energy of a system of particles is equal to the sum of the kinetic energies of all the particles

116a. For a rigid body which has a motion of translation, the kinetic energy is $E = \frac{1}{2}(W/g)v^2$, where W is the weight of the body and v is the velocity of any point in the body.

116b. For a rigid body rotating about a fixed axis, the kinetic energy is $E = \frac{1}{2}I_C\omega^2$, where I_C is the moment of inertia of the body about the axis of rotation, and ω is the angular velocity of the body.

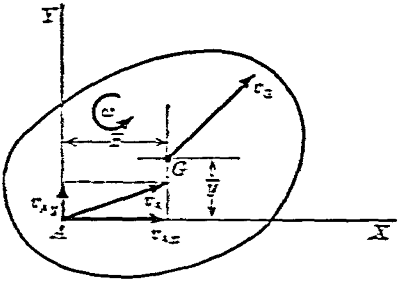


FIG. 44

117. For a body having a plane motion, the kinetic energy is (Fig. 44): $E = \frac{1}{2}(W/g)v_A^2 + \frac{1}{2}I_A\omega^2 + (W/g)\omega(\bar{x}v_{Ay} - \bar{y}v_{Ax})$, where v_A is the velocity of base point A, ω is the angular velocity of the body, and I_A is the moment of inertia of the body about an axis through A.

117a. Referring the motion of the body to its center of gravity G, we may write the kinetic energy of a body with plane motion in the form

$$E = \frac{1}{2}\frac{W}{g}\bar{v}^2 + \frac{1}{2}\bar{I}_G\omega^2.$$

117b. If the instantaneous center of rotation I is taken as the base point, the expression for the kinetic energy of the body reduces to $E = \frac{1}{2}I_I\omega^2$, where I_I is the moment of inertia of the body about an axis passing through the instantaneous center.

Principle of Work and Kinetic Energy.

118. For a particle moving along any path from position A to position B, the change in kinetic energy of the particle from A to B is equal to the work done by all forces acting on the particle during this change of position: $\frac{1}{2}m(v_B^2 - v_A^2) = \Sigma \int_A^B F \cos\phi \, ds.$

119. The change in kinetic energy E of a system of particles for any period of time is equal to the work done by all forces acting on the particles during that interval of time.

119a. For all rigid bodies (and systems of particles in which the work done by the internal forces is zero) the change in kinetic energy during any motion is equal to the work done by the external forces acting on the body: $E_2 - E_1 = \Sigma W_i$, where E_2 and E_1 are the final and initial kinetic energies, and W_i is the work done by any external force F_i during the motion.

IMPULSE AND MOMENTUM

Impulse of a Force

120 The impulse of a force F in the time interval dt is the product $F \times dt$ The impulse is a vector quantity whose direction is that of the force F

120a If the force F is constant in magnitude and direction, its *impulse for any time interval $t_2 - t_1$ is equal to $F\ (t_2 - t_1)$*

120b The component of the impulse of a force F in the direction XX during any time interval $t_2 - t_1$ is equal to $\int_{t_1}^{t_2} F_x\,dt$ When F_x is constant, this becomes $F_x\ (t_2 - t_1)$

Angular Impulse

121 When a force F exerts a moment M about an axis NN, its angular impulse about this axis during the time interval from t_1 to t_2 is $\int_{t_1}^{t_2} M\,dt$

121a If M is constant, the angular impulse for any time interval $t_2 - t_1$ is equal to $M\ (t_2 - t_1)$

122 The impulse of a system of forces in any direction XX is the algebraic sum of the x components of the impulses of all the forces of the system If the forces vary, the x component of the impulse for a time interval $t_2 - t_1$ is $\Sigma \int_{t_1}^{t_2} F_x\,dt$ If the forces are constant during the interval, the x component of the impulse of the force system is $(\Sigma F_x)(t_2 - t_1)$

122a The angular impulse of a system of forces about any axis NN during the time interval $(t_2 - t_1)$ is $\Sigma \int_{t_1}^{t_2} M_i\,dt$, where M_i is the moment of any force F_i about axis NN If the moment ΣM_i of all forces about the axis is constant during the time interval, the angular impulse is $(\Sigma M_i)(t_2 - t_1)$

Momentum

123 The momentum of a particle of mass m, moving with a velocity v, is equal to mv The momentum is represented by a vector passing through the particle and having the direction of the velocity and a magnitude mv

Moment of Momentum

124 The product of the component of the momentum (Fig 45) of a particle P lying in a plane KK normal to the axis NN, and the per

pendicular distance $CP = ED = d$, between the vector of the momentum and the axis NN, is the moment of momentum H_n of the particle with respect to axis NN.

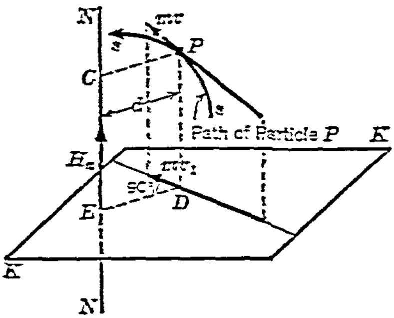


FIG. 45

The moment of momentum can be represented by a vector H_n whose *direction is parallel to axis NN* and whose magnitude is equal to mv_1d. The sense of the vector H_{n_1} is chosen such that the moment of the momentum is clockwise when viewed in the direction in which the vector points.

Momentum of a System of Particles.

125. The momentum of a system of particles is the vector sum of the momenta of all the particles. The momentum U of the system is equal to the product of the mass W/g of the whole system and the velocity $\bar{v}$ of the mass center of the system. Its direction is the same as that of the velocity of the mass center. We may write

$$U = \frac{W}{g}\bar{v}.$$

125a. The component of the momentum of a system of particles parallel to any axis xx is $U_x = (W/g)\bar{v}_x$, where $\bar{v}_x$ is the x component of the velocity of the mass center.

Angular Momentum.

126. The sum of the moments of momentum of all particles of a system with respect to an axis NN is the angular momentum of the system with respect to the axis.

126a. The angular momentum H_n of a rigid body rotating about a fixed axis NN is equal to the product of the moment of inertia I_n of the body about axis NN and the angular velocity ω of the body: $H_n = I_n\omega$. The vector of the angular momentum is parallel to the axis of rotation.

127. For a rigid body which has a plane motion (Fig. 44), the angular momentum H_A with respect to the axis passing through a base point A and normal to the plane of motion is

$$H_A = I_A\omega + \frac{W}{g}\cdot v_{Ay}\cdot\bar{x} - \frac{W}{g}\cdot v_{Ax}\cdot\bar{y},$$

where I_A is the moment of inertia of the body about the axis through A, and ω is the angular velocity of the body.

127a The angular momentum of a rigid body with respect to an axis passing through the center of gravity of the body is $H_G = \bar{I}_G\omega$

127b The angular momentum of a body with respect to an axis passing through the instantaneous center I of the body is $H_I = I_I\omega$, where I_I is the moment of inertia of the body about the axis passing through the instantaneous center

Principle of Impulse and Momentum

128 During any time interval, the change in the component of the momentum of a particle parallel to any axis XX is equal to the x component of the impulse of all forces acting on the particle during the interval

$$U_{2x} - U_{1x} = m(v_{2x} - v_{1x}) = \Sigma \int_{t_1}^{t_2} F_x \, dt$$

128a The change in the component of momentum of a rigid body or a system of particles, parallel to any axis xx, during a time interval $t_2 - t_1$, is equal to the x component of the impulse of all external forces acting on the body or system of particles during the time interval

$$U_{2x} - U_{1x} = \frac{W}{g} \ (v_{2x} - v_{1x}) = \Sigma \int_{t_1}^{t_2} F_x \, dt$$

129 If the resultant of all external forces acting on a system of particles has a component in any direction xx equal to zero, the x component of the momentum of the system remains constant

130 When a particle moves under the action of forces, its motion is such that at any instant the time rate of change of the moment of momentum dH_n/dt, with respect to any axis NN, is equal to the moment M_n of the forces with respect to that axis

$$\frac{dH_n}{dt} = M_n$$

130a During any time interval $t_2 - t_1$, the change in angular momentum of a particle with respect to any axis NN is equal to the angular impulse about the axis of all forces acting on the particle during the interval $H_{n_2} - H_{n_1} = \int_{t_1}^{t_2} M_n \, dt$

131 The time rate of change of the angular momentum of a rigid body or any other system of particles with respect to an axis NN is equal to the moment about the axis of all external forces acting on the system $dH_n/dt = M_n$ If M_n is equal to zero, the angular momentum H_n remains constant

131a. During any time interval $t_2 - t_1$, the change in angular momentum of a body rotating about a fixed axis is equal to the angular impulse of all external forces about the axis of rotation during the interval: $H_{z_2} - H_{z_1} = \int_{t_1}^{t_2} M_z \, dt$.

131b. For a body which has a plane motion, the change in angular momentum of the body about an axis normal to the plane of motion and passing through the center of gravity of the body, during a time interval $t_2 - t_1$, is equal to the angular impulse of all external forces about that axis during the interval:

$$H_{G_2} - H_{G_1} = \int_{t_1}^{t_2} M_G \, dt.$$

MOTION OF THE CENTER OF GRAVITY

132. For any system of particles under the action of a group of forces, the product of the mass and the component of acceleration $\bar{a}_x$ of the mass center parallel to any axis XX is equal to the algebraic sum of the x components of all external forces acting on the body: $\Sigma F_x = (W/g)\bar{a}_x$.

132a. If the forces applied to a system of particles are in equilibrium or result in a couple, the center of gravity of the system moves with a constant velocity or remains at rest.

132b. The motion of the center of gravity of a system of particles does not change when the internal forces of the system vary. Its state of motion is not affected when internal forces are created or disappear, as occurs when parts of the system collide or explode.

BEARING REACTIONS

133. A rigid body, rotating with angular velocity ω and angular acceleration α about a fixed axis defined by two bearings, generally exerts forces on these bearings (Fig. 46). The reactions of the bearings on the axis and other external forces acting on the body are in equilibrium with the system of inertia forces of the body. Using the axis of rotation as the z axis in an x, y, z coordinate system, we may say that the resultant inertia force system is completely

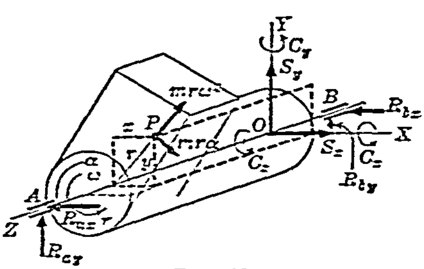


FIG. 46

specified by the following components

$$S_x = \frac{W}{g} x\omega^2 + \frac{W}{g} \bar{y}\alpha,$$

$$S_y = \frac{W}{g} \bar{y}\omega^2 - \frac{W}{g} x\alpha,$$

$$C_x = \alpha \int_M xz\, dm - \omega^2 \int_M yz\, dm = \alpha P_{xz} - \omega^2 P_{yz},$$

$$C_y = \omega^2 \int xz\, dm + \alpha \int yz\, dm = \omega^2 P_{xz} + \alpha P_{yz},$$

$$C_z = I_z \alpha,$$

where S_x and S_y are inertia force components parallel to the x and y axes, and C_x, C_y, and C_z are the moments produced by all inertia forces about the x, y and z axes, respectively P_{xz} is the product of inertia of the body with respect to the x and z axes and P_{yz} with respect to the y and z axes

133a If the coordinate axes are chosen so that the ZOY plane passes through the center of gravity of the body ($\bar{y} = 0$) the inertia force components are $S_x = (W/g)x\omega^2$, $S_y = (W/g)x\alpha$, while C_x, C_y and C_z remain the same as above

134 When $P_{xz} = P_{yz} = 0$ i e , when the axis of rotation is a principal axis of inertia of the body, the inertia couples C_x and C_y are both zero This is the case when (*a*) the axis of rotation is normal to a plane of symmetry which is chosen as the XOY plane, (*b*) the body has a line of symmetry parallel to the axis of rotation and the XOY plane is chosen through the center of gravity of the body

134a Computation of the resultant inertia force system can be *simplified by dividing the body into several parts for each of which the* products of inertia $P_{x_1z_1}$ and $P_{y_1z_1}$ for coordinate axes X_1, Y_1, Z_1 in that body are zero

135 The bearing reactions will be unaffected by the rotation of the body when the center of gravity of the body is on the axis of rotation $x = \bar{y} = 0$ and when the axis of rotation is a principal axis of inertia

VIBRATIONS AND OSCILLATIONS

Free Harmonic Vibrations.

136 When a body Q is moving along the axis XY under the action of a restoring force F which is proportional to the distance x of the body from a fixed point O, and directed toward the point (Fig 47), the

equation of motion is

$$Ma = \frac{W}{g}\frac{d^2x}{dt^2} = -kx, \qquad \text{or} \qquad \frac{d^2x}{dt^2} = -\frac{kg}{W}x,$$

where k is the spring constant of the restoring device.

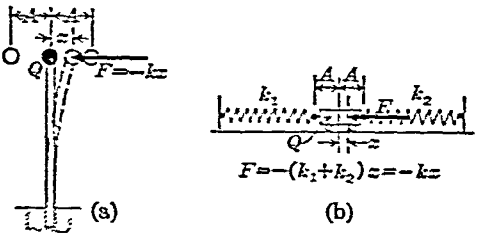


FIG. 47

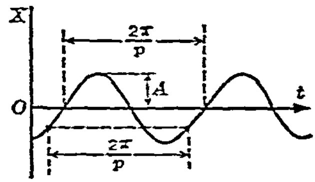


FIG. 48

136a. Integrating the equation of motion, $d^2x/dt^2 + p^2x = 0$, where $p^2 = kg/W$ (see § 85b), we obtain the position-time relation

$$x = A \sin (pt + D), \qquad \text{or} \qquad x = B \sin pt + C \cos pt.$$

The values of the constants A and D, or B and C, are determined from two known conditions of motion. The motion is harmonic with amplitude A (Fig. 48). The period of oscillation, or the time of one complete cycle of oscillation, is $T = 2\pi/p$, and the frequency of oscillation is $f = 1/T = p/(2\pi)$.

Damped Oscillations.

137. If a frictional resistance to motion exists, and the friction is assumed to be proportional to the velocity of the body, that is, if we have $F_r = -rv = -r\cdot dx/dt$, the force acting on the body at any instant is $F_x = -kx - r\cdot dx/dt$. The equation of motion is

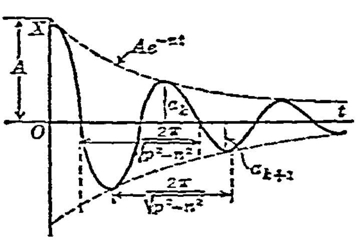


FIG. 49

$$\frac{W}{g}\frac{d^2x}{dt^2} = -kx - r\frac{dx}{dt},$$

or

$$\frac{d^2x}{dt^2} + 2n\frac{dx}{dt} + p^2x = 0,$$

where $p^2 = gk/W$ and $2n = gr/W$. Integration of this equation gives, for $p > n$ (Fig. 49),

$$x = Ae^{-nt} \sin (\sqrt{p^2 - n^2}\, t + D)$$
$$= e^{-nt}(B \sin \sqrt{p^2 - n^2}\, t + C \cos \sqrt{p^2 - n^2}\, t).$$

The constants A and D, or B and C, may be determined from two known

conditions of the motion The period of oscillation is $T = 2\pi/\sqrt{p^2 - n^2}$ and the frequency is $f = 1/T = \sqrt{p^2 - n^2}/(2\pi)$ The amplitude Ae^{-nt} decreases with time The ratio of any excursion A_{k+1} to the preceding one A_k (on the opposite side of the equilibrium position) is given by $A_{k+1}/A_k = e^{-nT/2}$

When p is smaller than n, i e, when the frictional resistance is relatively large, the body moves aperiodically to its equilibrium position The solution in this case is $x = Ae^{(-n+\sqrt{n^2-p^2})t} + De^{(-n-\sqrt{n^2-p^2})t}$ The values of the constants A and D are determined from two known conditions of the motion

Forced Oscillations without Damping

138 If, in addition to the restoring force $-kx$, an external periodic disturbing force acts on the body Q (Fig 47), it undergoes a forced oscillation If the external force is expressed by $F = b \sin qt$, where b is the maximum absolute value of the force and $q/(2\pi)$ is the frequency of the force variations, the equation of motion is $(W/g)d^2x/dt^2 = -kx + b \sin qt$, or $d^2x/dt^2 + p^2x = h \sin qt$, where $p^2 = gk/W$ and $h = gb/W$ When $q \neq p$, the integration of this equation gives $x = A \sin (pt + D) + [h/(q^2 - p^2)] \sin qt$ The motion consists of two harmonic oscillations of different frequencies superimposed on each other The first term represents the free oscillations of the body and the second term the forced oscillations The amplitude of the forced oscillation is $h/(p^2 - q^2)$, and its frequency is $q/(2\pi)$, equal to the frequency of the disturbing force In the case $q = p$, the condition of resonance exists, the integration of the equation of motion gives $x = B \sin (qt + C) + (h/2p)t \cos qt$ The amplitude of the forced oscillation increases indefinitely with time The constants A and D, or B and C, are determined from two known conditions of motion

Forced Oscillation with Damping

139 If the body Q (Fig 47) is acted upon by a periodic external force, $F_1 = b \sin qt$, and by a frictional damping force, $F_2 = -r\, dx/dt$, the equation of motion is $(W/g)(d^2x/dt^2) = -kx - r\, dx/dt + b \sin qt$ or $d^2x/dt^2 + 2n\, dx/dt + p^2x = h \sin qt$, where $p^2 = gk/W$, $2n = gr/W$, and $h = gb/W$ Integration gives

$$x = Ae^{-nt} \sin (\sqrt{p^2 - n^2}\, t + D) + \frac{h}{\sqrt{(p^2 - q^2)^2 + 4n^2q^2}} \sin (qt + \delta)$$

The motion of the body consists of two superimposed harmonic oscillations one, a transient oscillation, damped out with time, represented by the first term of the equation for x, and a sustained forced oscillation, represented by the second term. The angle δ is the phase difference

IMPACT

142 When two bodies in motion collide, they compress in the zone of contact If the material of the bodies is elastic, the internal forces thus created cause the bodies to separate and to move subsequently *with velocities different from their velocities before the collision* The total amount of impact experienced by either body is reckoned by the change in its momentum $(W/g)v$ caused by the collision, where W is the weight of the body, and v is its velocity The impact is measured by the vector difference between the velocities after and before the collision If we neglect the action of friction, this velocity variation is parallel to the normal to the surfaces at the contact point

143 *The motion of the common center of gravity of colliding bodies* does not undergo any change during the collision, notwithstanding the sudden change in motion of each individual body (§ 132b)

Collision of Two Smooth Spherical Bodies

144 Two balls undergo a direct impact (Fig 52) when their centers C and D move along the same straight line before collision The velocities are taken as positive in one direction along the line, and negative in the opposite direction

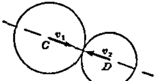


Fig 52

The total momentum of the two balls does not change during impact, hence we may write $(W_1/g)v_1 + (W_2/g)v_2 = (W_1/g)v_1' + (W_2/g)v_2'$, where v_1 and v_2 are the velocities of the balls before impact, and v_1' and v_2' after impact

If the material of the balls is completely inelastic, the balls deform and move together with a velocity $v_1' = v_2' = v'$ If the material of the balls is perfectly elastic, the kinetic energy after impact is equal to that before impact, or

$$\frac{1}{2}\frac{W_1}{g}v_1^2 + \frac{1}{2}\frac{W_2}{g}v_2^2 = \frac{1}{2}\frac{W_1}{g}(v_1')^2 + \frac{1}{2}\frac{W_2}{g}(v_2')^2$$

If the material of the balls is not perfectly elastic, then we have $v_2' - v_1' = -e(v_2 - v_1)$, where e is the coefficient of restitution which is known for various materials from experiments For completely inelastic bodies $e = 0$, and for perfectly elastic bodies, $e = 1$ The final values of the velocities of the two bodies after impact are obtained from the equation, $v_2' - v_1' = -e(v_2 - v_1)$ and the momentum equation $(W_1/g)(v_1 - v_1') = (W_2/g)(v_2' - v_2)$

Indirect Impact.

145. If the impact is indirect (Fig. 53), i.e., if the centers C and D of the balls do not move in the same straight line before collision, the velocities v_1 and v_2 before collision may be resolved into components v_{1o} and v_{2o} directed along the center line CD, and into components normal to the line CD. Their normal components v_{1n} and v_{2n} do not change during the collision; the components v_{1o}' and v_{2o}' in the line CD of the velocities after the collision are connected with the components v_{1o} and v_{2o} by the same equations as those which define the direct impact.

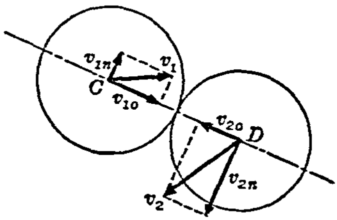


FIG. 53

Center of Percussion.

146. If a rigid body free to rotate around a fixed axis is struck by another body, the axis of rotation generally experiences an impact. The total moment of momentum of the two bodies with respect to the axis of rotation remains unchanged during the impact. The axis of rotation of a rigid body will not suffer any impact when the following conditions are fulfilled: (1) the line of the blow delivered to the body is normal to the plane through the axis of rotation and the center of gravity of the body; (2) a plane through this line, normal to the axis of rotation, intersects this axis in a point for which the axis is a principal axis of inertia; (3) the distance l from the line of the blow to the axis of rotation is $l = k_0^2/h$, where k_0 is the radius of gyration of the body with respect to its axis of rotation, and h is the distance from the center of gravity of the body to the axis of rotation. This point in the body where a blow does not produce an impact on the axis of rotation is called the center of percussion of the body; were the body considered as a pendulum, this point would be the center of oscillation.

PRINCIPLE OF VIRTUAL DISPLACEMENTS

147. The question of equilibrium of forces applied to a system of particles is often conveniently analyzed by use of the principle of virtual displacements. (Many problems given in the first part of the book may be solved by this method.)

A virtual displacement of a system of particles is an infinitely small possible displacement of the particles of the system, consistent with the geometrical constraints between the particles. If a force acts on the system of particles, the work done by the force during a virtual displacement of its point of application is the virtual work of the force.

148. Several forces $F_1, F_2, \cdots, F_n$ applied to a system of particles are in equilibrium if the sum of the virtual work done by all forces is zero for any virtual displacement of the system, i.e., when

$$\Sigma F_i \delta s_i \cos(\angle F_i, \delta s_i) = 0, \qquad \text{or} \qquad \Sigma(F_{x_i}\delta x_i + F_{y_i}\delta y_i + F_{z_i}\delta z_i) = 0,$$

where $F_i \delta s_i \cos(\angle F_i, \delta s_i)$ is the virtual work of any force F_i over the virtual displacement δs_i of its point of application, and F_{x_i}, F_{y_i}, F_{z_i} and δx_i, δy_i, δz_i are projections of the force F_i and the displacement δs_i on the coordinate axes.

149. The equation of motion of a system may be written by equating to zero the sum of the virtual work of all impressed forces and of the inertia forces of all particles of the system, over any arbitrary virtual displacement of the system.

PROBLEMS

PART I. STATICS

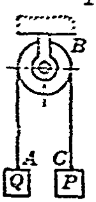
PLANE STATICS

1. Concurrent Forces.

1. A tug pulls three barges in a line. The propeller thrust is 3600 lbs. The water resistance to the tug is 1200 lbs., to the first barge 1200 lbs., to the second 800 lbs., and to the third 400 lbs. With a cable good for 400 lbs. maximum load, how many strands are necessary to connect the tug to the first barge, the first barge to the second, and the second to the third? *Ans.* 6, 3, 1 strands.

2. A man weighing 160 lbs. lifts a load of 120 lbs. by means of a rope passed over a fixed pulley. What is the force between the man's feet and the ground? What is the maximum load he can lift with this arrangement? *Ans.* 40 lbs.; 160 lbs.

3. A weight $Q = 60$ lbs. is balanced by a counterweight P. The rope ABC passing over a small pulley B is 30 feet long and weighs 10 lbs. Find the weight P and tensions at A, B, and C for the following conditions:



1. When A and C are at the same height.
2. When A is in its highest position.
3. When A is in its lowest position.

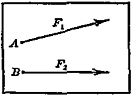
Ans.

	Weight P lbs.	Tension in rope, lbs. at A	B	C
1.	60	60	65	60
2.	50	60	60	50
3.	70	60	70	70

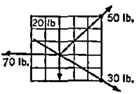
4. A train runs at constant speed on a level track. It weighs, excluding the weight of the locomotive, 360,000 lbs. What is the drawbar pull if the effective coefficient of friction is 0.005? *Ans.* 1800 lbs.

5. Two horses on opposite banks of a canal pull a barge moving parallel to the banks by means of two ropes. The tensions in these are 200 lbs. and 240 lbs. The angle between them is 60°. Find the pull on the barge and the angles α and β between the ropes and the banks of the canal.

Ans. Pull = 382 lbs.; $\alpha = 33°$; $\beta = 27°$.



6. Find by construction the size and direction of the resultant of two forces F_1 and F_2 when their intersecting point is outside the drawing limits.



7. Replace the force system shown by the simplest equivalent system.

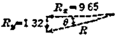
Solution:

The system reduces to a resultant R (§ 9):

$$R_x = \Sigma F_x, \qquad R_y = \Sigma F_y, \qquad R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}.$$

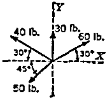
Force	F_x	F_y
20	—	$-$ 20.
30	+ 25	$-$ 16 67
50	+ 35 35	+ 35.35
70	$-$ 70 0	—
	$\Sigma F_x = -$ 9.65	$\Sigma F_y = -$ 1.32

$$R = \sqrt{(9.65)^2 + (1.32)^2} = 9.72 \text{ lbs.}$$



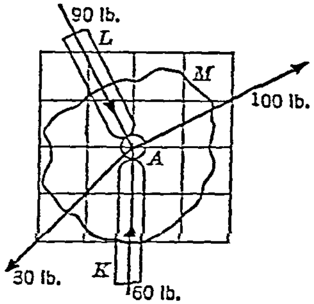
$$\theta = \sin^{-1}\frac{1\ 32}{9\ 72} = 7°48'$$

$$\theta_x = 187°48'$$



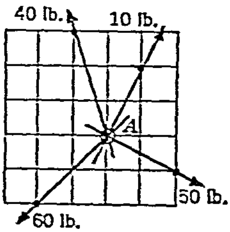
8. Determine the resultant of these four forces. (*a*) Using algebraic methods. (*b*) Using graphical methods.

Ans. $R = 48.1$ lbs.; $\theta_x = 112°$.



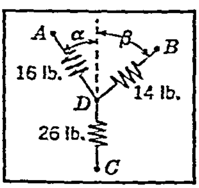
9. Two cables are attached to the boulder M at the ring A and exert tensions as indicated; while two thrust poles K and L exert forces as indicated. Find the resultant of these four applied forces.

Ans. $R = 108.4$ lbs.; $\theta_x = 1°28'$.



10. Four tension wires are attached to the head of a post as shown at A. Their directions and tensions are indicated. Replace the four wires by a single one that will produce an equivalent pull on the post.

Ans. $R = 18.75$ lbs.; $\theta_x = 251°40'$.



11. The rings A, B and C of three spring balances are fixed on a horizontal board. Three threads connected at D are tied to the hooks of the balances. The scales read 16, 14 and 26 lbs. Find the angles α and β as shown on the sketch. *Ans.* $\alpha = 27.7°$; $\beta = 32.2°$.

12. A board is tilted to make an angle α with the horizontal such that a heavy body on its surface slides downward with the constant velocity with which it is started. *Find the coefficient* of friction f (f equals the ratio between friction and normal forces on the board). *Ans.* $f = \tan \alpha$.

13. A railway car weighing 20,000 lbs. coasts down an 0.8% grade and reaches a constant maximum velocity after a certain time. What is the frictional resistance?

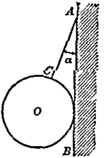
NOTE: % grade = tangent of the slope angle multiplied by 100.

Ans. 160 lbs.

14. A train moves at constant speed up an 0.8% grade. The cars weigh 760,000 lbs. What is the drawbar pull if the overall coefficient of friction is 0.005? *Ans.* $F = 9880$ lbs.

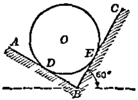
15. A 20-lb. ball is held on an inclined plane by a rope attached to a spring balance. The balance reads 10 lbs. The angle of

inclination of the plane with the horizontal is 30° Find the angle α between the rope and the vertical and the force Q exerted by the ball on the plane *Ans* $\alpha = 60°$, $Q = N = 17\ 3$ lbs

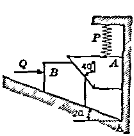


16 A ball O suspended on the string AC rests against the smooth vertical wall AB The angle BAC is α The weight of the ball is W Find the tension T of the rope and the force Q between the ball and the wall

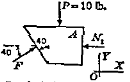
$$Ans \quad T = \frac{W}{\cos\alpha}, \quad Q = W \tan\alpha$$



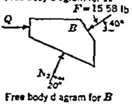
17 A 12-lb ball O lies between mutually perpendicular smooth planes AB and BC Find the force against each surface if plane BC is inclined 60° to the horizontal *Ans* $N_D = 10\ 4$ lbs , $N_E = 6$ lbs



18 In an instrument, blocks A and B slide over the sides of an angle K as shown Spring P exerts a downward force of *10 lbs* on block A Neglecting the weight of the blocks and frictional effects (assuming all contact surfaces to be smooth), find the force Q necessary to preserve the equilibrium of the blocks



Free body diagram for A



Free body diagram for B

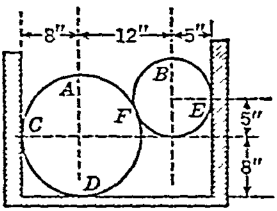
Solution

Body A is in equilibrium under the action of all forces acting on it (Art 10)

$$\Sigma F_x = 0 = F \cos 40° - N_1,$$
$$\Sigma F_y = 0 = F \sin 40° - 10,$$
$$F = 15\ 58 \text{ lbs},$$
$$N_1 = 12.0 \text{ lbs}$$

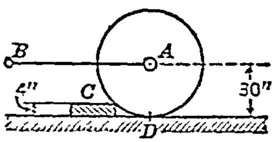
Body B is in equilibrium (Art 10)

$$\Sigma F_x = 0 = Q + N_2 \sin 20° - 15\ 58 \cos 40°,$$
$$\Sigma F_y = 0 = N_2 \cos 20° - 15\ 58 \sin 40°$$
$$Q = 8.3 \text{ lbs}$$
$$N_2 = 10\ 6 \text{ lbs}$$

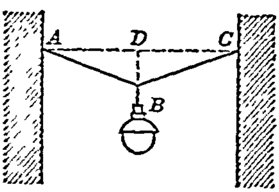


19. Two smooth cylinders A and B are placed in a box as shown. Cylinder A weighs 40 lbs. and B weighs 30 lbs. The diameter of A is 16 in., the diameter of B is 10 in. Find the reactions at C, D, E and F. (*a*) Solve algebraically. (*b*) Solve graphically.

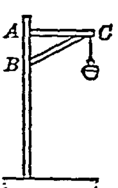
Ans. $F_E = F_C = 72$ lbs.; $F_F = 78$ lbs.; $F_D = 70$ lbs.



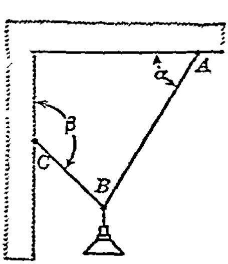
20. A roller 60 in. in diameter weighs 4000 lbs. What is the horizontal force P on handle AB necessary to pull the roller over a stone 4 in. high? *Ans.* $P \geqq 2300$ lbs.



21. A 30-lb. arc lamp hangs in the middle of a 60-ft. cable ABC suspended from two hooks at A and C, both on the same level. Find the tension in each side of the cable if the sag BD at the lamp is 0.3 ft. *Ans.* $T_C = T_A = 1500$ lbs.

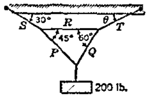


22. A 60-lb. arc lamp is suspended from a vertical post by means of a horizontal cross bar AC 4 ft. long and a brace BC 5 ft. long. What are the forces S_1 and S_2 in AC and BC? Show the directions of the forces by denoting tension as positive and compression as negative. *Ans.* $S_1 = 80$ lbs.; $S_2 = -100$ lbs.



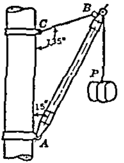
23. A 4-lb. electric lamp is suspended from the ceiling by means of a cord AB. The lamp is pulled towards a vertical wall by a string BC. The cord AB makes an angle $\alpha = 60°$ with the ceiling and the string BC is inclined at an angle $\beta = 135°$ to the wall. What are the tensions in the cord and string?

Ans. $T_A = 2.93$ lbs.; $T_C = 2.07$ lbs.



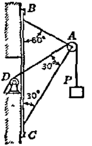
24. A 200-lb. weight is suspended by two cords. A horizontal tie R holds the cords in the positions shown. Determine the tensions in the four cords P, S, Q and T and in the tie R. Also determine the angle θ.

Ans. $P = 103.5$ lbs.; $Q = 146.5$ lbs.; $S = 146.5$ lbs.; $R = 53.75$ lbs.; $T = 179.5$ lbs.; $\theta = 45°$.



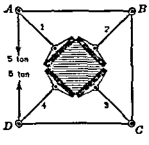
25. A derrick consisting of a boom AB hinged to the mast at A and a chain CB carries a load $P = 400$ lbs. suspended from B. Angle $BAC = 15°$ and angle $ACB = 135°$. Find the tension T in the chain and the compression Q in the boom.

Ans. $T = 207$ lbs.; $Q = 564$ lbs.



26. A wall crane BAC lifts a 4000-lb. load by means of a chain on pulleys at A and D. Angle $CAD = 30°$, $ABC = 60°$, $ACB = 30°$. Find the forces Q_1 in AB and Q_2 in AC.

Ans. $Q_1 = 0$; $Q_2 = -6930$ lbs.



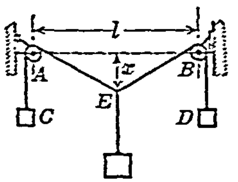
27. The following mechanism is used to compress a small cement cube on four faces. Links AB, BC, CD are the sides of a square $ABCD$, while the links 1, 2, 3 and 4 are of equal length and are directed along the diagonals of the square; all connections are hinged. Two equal and opposite forces P are applied to points A and D. Find the forces N_1, N_2, N_3 and N_4 compressing the cube and the tensions S_1, S_2 and S_3 in the links AB, BC and CD, if P is 5 tons.

Ans. $S_1 = S_2 = S_3 = P = 5$ tons; $N_1 = N_2 = N_3 = N_4 = 7.07$ tons.

28. A rectangular plate weighing 10 lbs. is suspended from hinges on its upper edge. A wind of uniform velocity impinging on the plate keeps it at an angle of 18° to the vertical. Find the normal force of the wind on the plate.

Ans. $Q = P \sin \alpha = 3.09$ lbs.

29. A rope $CAEBD$ is passed over two negligibly small pulleys A and B mounted on the same level. The distance between A and B is l. Two equal weights w are attached to C and D and a load W is suspended at E. Under conditions of equilibrium what is the distance x between E and line AB?

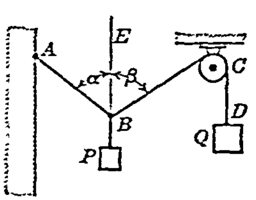


Ans. $x = \dfrac{Wl}{2\sqrt{4w^2 - W^2}}$.

30. A weight of 25 lbs. is held by two ropes which pass over two pulleys. Counterweights are suspended on the free ends of the ropes. One of these weighs 20 lbs. and the sine of the angle which its rope makes with the vertical is 0.6. Find the other weight p and the angle β which its rope makes with the vertical.

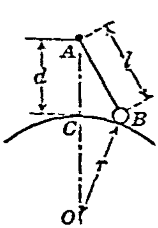
Ans. $p = 15$ lbs.; $\beta = \tan^{-1} 4/3$.

31. One end of the rope AB is fixed to a wall at A. A weight P and another rope BCD passed over a pulley at C are attached to the other end. A weight $Q = 20$ lbs. is suspended at D. The system is in equilibrium when the angles between the ropes and the vertical BE are $\alpha = 45°$ and $\beta = 60°$. What is the weight P and the tension T in the rope AB?

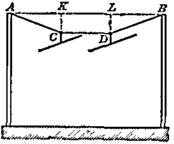


Ans. $P = 27.3$ lbs.; $T = 24.5$ lbs.

32. A small ball B of weight W is suspended by a thread AB from a fixed point A. It rests on the surface of a smooth sphere whose radius is r. The distance $AC = d$. The length of the thread $AB = l$. AO is vertical. Find the tension T in the thread and the reaction Q of the sphere.



Ans. $T = W\dfrac{l}{d + r}$; $Q = W\dfrac{r}{d + r}$.

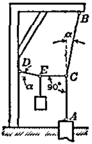


33. Two trolley wires are suspended on cross cables stretched between two posts. The cross cables are spaced 120 ft. apart.

$$AK = KL = LB = 15 \text{ ft.}; \quad KC = LD = 1.5 \text{ ft.}$$

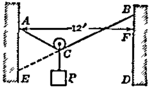
Neglecting the weight of the cross cables, *find the tensions* T_1, T_2 *and* T_3 in the parts AC, CD and DB if the trolley wire weighs ½ lb. per ft.

Ans. $T_1 = T_3 = 603$ lbs.; $T_2 = 600$ lbs.



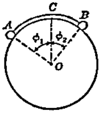
34. A workman attempting to pull a pile out of the ground tied a cable to it at A. He fixed the other end of the cable at B, attached another cable to the point C and fixed this cable at D. Then he exerted a pull of 200 lbs. on the second cable at E. Before the pile began to move AC was vertical, EC was horizontal, BC made an angle of 4° with the vertical and DE made an angle of 4° with the horizontal. What was the tension in AC?

Ans. $T = 40{,}900$ lbs.



35. A pulley C carrying a weight $P = 36$ lbs. can slide along a flexible cable ACB hung between two walls AE and BD. The distance between them is 12 ft., the length of the cable is 15 ft. Find the tension in the cable, neglecting the effects of its own weight.

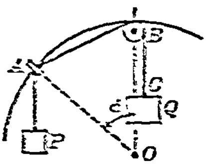
Ans. $T = 30$ lbs.



36. Two small balls A and B rest on a circular cylinder of radius $OA = 3''$, whose axis is horizontal. A weighs 2 oz. and B weighs 4 oz. The balls are connected by a thread 6 in. long. Find the angles ϕ_1 and ϕ_2 between the radii OA and OB and the vertical OC, and the forces N_1 and N_2 of the balls against the cylinder at equilibrium.

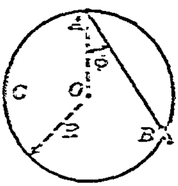
Ans. $\phi_1 = 84° 45'$; $\phi_2 = 29° 50'$; $N_1 = 0.18$ oz.; $N_2 = 3.47$ oz.

37. A smooth ring slides without friction on a rod bent into a circular arc whose plane is vertical. A weight P is tied to the ring. A rope ABC is also attached to the ring and passed over a pulley suspended from the highest point of the rod and at its end C a weight Q is suspended. Find the angle ϕ subtended by arc AB when the system is in equilibrium. Consider the ring as weightless.



Ans. $\phi = 2 \sin^{-1} [Q/(2P)]$.

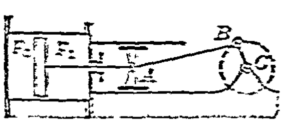
38. A smooth ring B of weight P can slide on a circular rod ABC whose plane is vertical. The ring is attached to A by means of an elastic string AB. The tension T of the string is k times the unit elongation. Find the angle ϕ when the system is in equilibrium.



NOTE: If l is the original length and L is the stretched length of the string, $T = k\cdot(L - l)/l$.

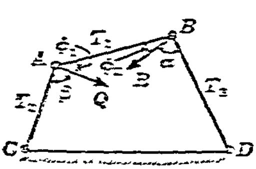
$$\textit{Ans.} \quad \cos\phi = \frac{1}{2}\cdot\frac{kl}{kR - Pl}.$$

39. The area of the piston in a steam engine is 125 sq. in. The connecting rod AB is 6 ft. long and the crank radius BC is 1.2 ft. The steam pressure is $p_0 = 95$ lbs./sq. in. and the back pressure is $p_1 = 15$ lbs./sq. in. Find the tangential force P acting on the crank and the force N between the cross-head and the guide when the angle $ABC = 90°$. Neglect the effects of friction.

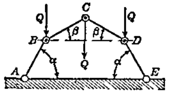


Ans. $P = 10{,}200$ lbs.; $N = 2000$ lbs.

40. $ABCD$ is a system of links. A force $Q = 20$ lbs. acts at A in a direction such that angle $BAQ = 45°$. Find the value of the force R acting at B which keeps the system in equilibrium. Angle $ABR = 30°$, $CAQ = 90°$, and $DBR = 60°$.

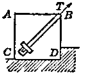


Ans. $R = 32.6$ lbs.

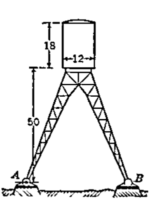


41. Four rods of equal length form a linkage A and E are fixed pivots The joints B, C and D are loaded with equal vertical weights Q At equilibrium rods AB and ED form an angle $\alpha = 60°$ with the horizontal What is the angle between rods BC and DC and the horizontal? *Ans* $\beta = 30°$

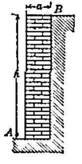
42 The cable of a suspension bridge is anchored in a block of masonry of square vertical cross section $ABCD$, 16 ft on a side



The specific gravity of the masonry is 2 5 The cable is built in along the diagonal CB and has a tension of 200,000 lbs What must be the third dimension a of the block to resist tipping over the edge D, *the action of the surrounding earth being neglected* *Ans* $a \geqq 7\ 08$ ft



43 A cylindrical water tank 12 ft in diameter and 18 ft high is mounted on four legs The bottom of the tank is 50 ft above the ground The complete structure weighs 16,000 lbs The wind pressure is calculated on the basis of 0 18 lbs /sq in on the vertical projected area of the tank Find the distance AB necessary to make the structure stable against the horizontal thrust of the wind *Ans* $AB \geqq 41\ 3$ ft

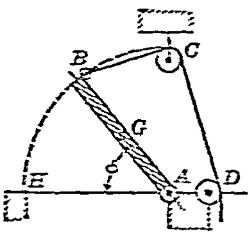


44 A vertical stone retaining wall is 15 ft high It has a specific gravity of 2 The horizontal thrust of the earth per running foot is 4,000 lbs and it is assumed to be acting at a point ⅓ of the distance from the bottom of the wall What width a of the wall is necessary to keep it from being tipped over the edge A? *Ans* $a \geqq 4\ 6$ ft

45 A point M is attracted to three immovable points $M_1(x_1, y_1)$, $M_2(x_2, y_2)$, $M_3(x_3, y_3)$ by forces proportional to the distances $F_1 = k_1 MM_1$, $F_2 = k_2 MM_2$, $F_3 = k_3 MM_3$, where k_1, k_2, and k_3 are

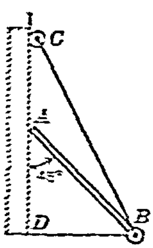
constants of proportionality. Find the coordinates x and y of M when the system is in equilibrium.

$$\textit{Ans.}\quad x = \frac{k_1x_1 + k_2x_2 + k_3x_3}{k_1 + k_2 + k_3}, \qquad y = \frac{k_1y_1 + k_2y_2 + k_3y_3}{k_1 + k_2 + k_3}.$$

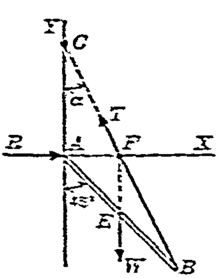


46. A 200-lb. skylight AB is hinged at A and is lifted by a rope BCD passing over pulleys C and D. C and A are on the same vertical line and $AB = AC$. Neglecting friction and considering the weight of the skylight concentrated at the center of gravity G, find the tension T in the rope as a function of the angle ϕ between AB and the horizontal line AH. What are the maximum and minimum tensions?

$$\begin{aligned} \textit{Ans.}\quad T &= 200 \sin (45° - \phi/2) \text{ lbs.},\\ T_{min} &= 0 \text{ at } \phi = 90°,\\ T_{max} &= 141 \text{ lbs. at } \phi = 0°. \end{aligned}$$



47. The upper end of a 6-ft. rod AB weighing 5 lbs. rests against a smooth vertical wall. A rope BC is attached to the lower end B and fixed to the wall at C in such a way that the rod forms an angle $BAD = 45°$ with the wall at equilibrium. Find the length AC, the tension T in the rope and the reaction R against the wall.



Solution:

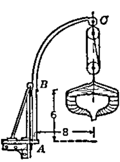
Three forces act on the body AB, its weight W, the wall reaction R, and rope tension T; W is parallel to the wall, while R is normal to the wall. The three non-parallel forces are in equilibrium; therefore they are concurrent at point F (§ 10a).

From geometrical considerations, $AC = 2FE = AE\sqrt{2} = 4.24$ *ft.*; $\alpha = \tan^{-1} 0.5 = 26° 34'$.

Considering the X and Y components of the forces,

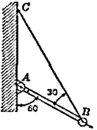
$$\Sigma F_x = R - T \sin \alpha = 0, \qquad \Sigma F_y = T \cos \alpha - W = 0.$$

$$T = \frac{W}{\cos \alpha} = 5.6 \textit{ lbs.}, \qquad R = W \tan \alpha = 2.5 \textit{ lbs.}$$



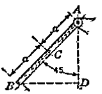
48 A 2400-lb boat hangs on two davits each carrying half the load The davit ABC rests in a ball and socket joint at its lower end A and passes through a bearing B 6 ft above A The span of the davit is 8 ft Neglecting the weight of the davit find the forces acting at A and B

Ans $A_y = 1200$ lbs , $A_x = 1600$ lbs , $B_x = 1600$ lbs



49 A 4-lb rod AB is hinged to a vertical wall at A It is held at an angle of 60° to the wall by a rope BC which forms an angle of 30° with the rod Find the magnitude and the direction of the reaction R of the hinge

Ans $R = 2$ lbs , $(\angle R, AC) = 60°$



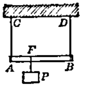
50 A skylight AB weighing 178 lbs rotates about an axis through A and rests on the roof at B $AD = BD$ Find the reactions of the supports, assuming the weight to be concentrated at the center C *Ans* $R_A = 141$ lbs , $R_B = 63$ lbs

2 Parallel Forces

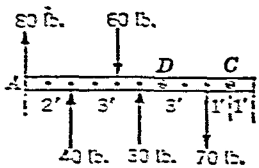
51 A beam of length l carrying a uniformly distributed load of p lbs per unit length rests on two end supports What are the reactions of the supports? *Ans* $R_1 = R_2 = \frac{1}{2}pl$ lbs

52 A beam of length l supported at both ends carries a concentrated load P lbs at a distance x from the left hand support What are the reactions of the supports?

Ans $R_1 = P\dfrac{l - x}{l}$ (left-hand), $R_2 = P\dfrac{x}{l}$



53 A 3 ft uniform rod AB weighing 4 lbs is suspended horizontally on two parallel strings AC and BD A weight $P = 24$ lbs is attached to the rod at E $AE = 3/4$ ft Find the tensions T in the strings *Ans* $T_B = 8$ lbs $T_A = 20$ lbs

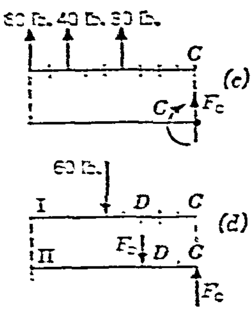


54. (*a*) Use algebraic methods to determine the resultant of the five parallel forces shown.

(*b*) Determine the resultant using a graphical method.

(*c*) What force applied at point C and couple are equivalent to the three forces acting upward?

(*d*) Replace the 60-lb. force by two parallel components applied at C and D.



Solution:

(*a*) The resultant R is (§ 15):

$$R = \Sigma F = 80 + 40 + 30 - 70 - 60$$
$$= +\ \mathit{20\ lbs.}\ \text{(acting upwards)}$$
$$\Sigma M_A = +40 \times 2 - 60 \times 4 + 30 \times 5 - 70 \times 8$$
$$= -\ 570\ \text{lbs. ft.}$$

A 20-lb. force acting upward to produce a 570 lbs. ft. clockwise moment about the point A would have to be 28.5 ft. to the left of A.

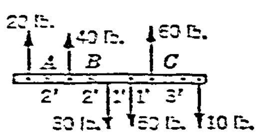
(*c*) For the two force systems to be equivalent (§ 20), we must have

$$F_C = 80 + 40 + 30 = \mathit{150\ lbs.}$$
$$\Sigma M_C = C = +30 \times 4 + 40 \times 7 + 80 \times 9 = \mathit{1120\ lb.\ ft.}\ \text{clockwise.}$$

(*d*):

$$M_D' = M_D'';\quad 60 \times 2 = 3 \times F_C;\quad F_C = \mathit{40\ lbs.}$$
$$M_C' = M_C'';\quad 60 \times 5 = F_D \times 3;\quad F_D = \mathit{100\ lbs.}$$

55. (*a*) Use algebraic methods to determine the resultant of the six parallel forces shown.



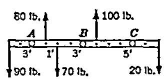
(*b*) Determine the resultant using graphical methods.

(*c*) What force applied at point C and couple are equivalent to the 40-lb. force acting upward?

(*d*) Replace the 20-lb. force by two components applied at points A and B; applied at points B and C.

Ans. (*a*) R = 30 lbs. up; 1 ft. 8 in. to the left of A.
(*c*) F_C = 40 lbs. up; 200 lbs.-ft. clockwise.
(*d*) F_A = 30 lbs. up; F_B = 10 lbs. down;
F_B = 35 lbs. up; F_C = 15 lbs. down.

56. (*a*) Using algebraic methods, determine the resultant of the five parallel forces shown.

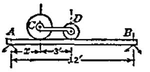


(*b*) Determine the resultant using a graphical solution.

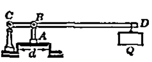
(*c*) What force applied at point A and couple are equivalent to the three forces acting downward?

(*d*) Replace the 80-lb. force by two components acting through points A and B; acting through B and C.

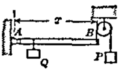
Ans. (*a*) Couple, 420 lbs.-ft. counterclockwise.
(*c*) 180 lbs. down; 160 lbs.-ft. clockwise.
(*d*) $F_A = 60$ lbs. up; $F_B = 20$ lbs. up.
$F_B = 140$ lbs. up; $F_C = 60$ lbs. down.



57. Two loads $C = 400$ lbs. and $D = 200$ lbs. rest on a horizontal beam 12 ft. long which is supported at A and B. The distance between loads is 3 ft. If the reaction at A is twice the reaction at B, what is the distance between A and the load C? *Ans.* $x = 3$ ft.

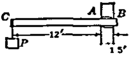


58. A safety valve A of a boiler is $2\frac{1}{2}$ in. in diameter. It is connected by a link AB to a 2-lb. lever CD which is 1.5 ft. long. The distance from B to the fulcrum C is 3 inches. If the valve is to open at 165 lbs./sq. in. pressure, what weight Q should be hung at D? *Ans.* $Q = 134$ lbs.



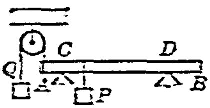
59. A horizontal rod weighing 100 lbs. is hinged at A. A lifting force of 150 lbs. is applied at the other end B by means of a weight P suspended by a rope over a pulley. A weight $Q = 500$ lbs. is hung 20 inches from B. The system is in equilibrium. How long is the rod?

Ans. $x = 25$ inches.

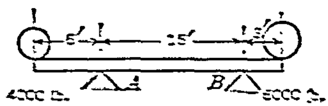


60. An iron beam 12 ft. long and weighing 1000 lbs. is built into a wall $1\frac{1}{2}$ ft. thick so that it rests against points A and B. A load $P = 8000$ lbs. is carried at the free end of the beam. What are the reactions at A and B?

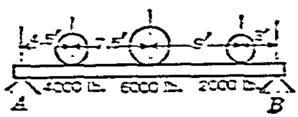
Ans. $R_A = 77{,}000$ lbs.; $R_B = 68{,}000$ lbs.



61. A beam 30 ft. long and weighing 400 lbs. rests on two supports C and D, 15 ft. apart. An upward force $Q = 600$ lbs. acts at a point $A = 6$ ft. from C. A weight $P = 1600$ lbs. is suspended at a point 3 ft. to the right of C. Find the reactions of the supports. *Ans.* $R_D = 800$ lbs.; $R_C = 600$ lbs.

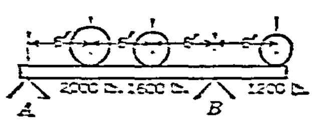


62. A beam 24 ft. long supported at two points 15 ft. apart carries two loads 4000 lbs. and 6000 lbs. one at each end. Dimensions are given in the sketch. Find the reactions of the supports, neglecting the weight of the beam. *Ans.* $R_A = 4400$ lbs.; $R_B = 5600$ lbs.



63. A beam 24 ft. long supported at both ends carries three loads, 4000 lbs., 6000 lbs., and 2000 lbs. located as shown in the sketch. Find the reactions of the supports.

Ans. $R_A = 6500$ lbs.; $R_B = 5500$ lbs.

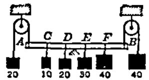


64. A beam 24 ft. long is supported at one end and at a point 18 ft. from that end. Three loads 2000 lbs., 1600 lbs., and 1200 lbs. are placed as shown in the sketch. Find the reactions of the supports, using a graphical method. *Ans.* $R_A = 1450$ lbs.; $R_B = 3350$ lbs.

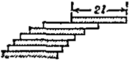
65. A beam 20 ft. long and weighing 640 lbs. is hinged to a wall and rests horizontally on a support 8 ft. from the wall. A load of 320 lbs. is applied at a point 6 ft. from the hinge and another load of 640 lbs. is applied at a point 14 ft. from the hinge. Find the reactions of the supports, using a graphical method.

Ans. $R_B = 2160$ lbs. up; $R_A = 560$ lbs. down.

66. A rod 12 ft. long weighing 12 lbs. carries four loads spaced 4 ft. apart. The loads from left to right are 4, 6, 8, and 10 lbs. How far from the left end should a single support be placed if the rod is to remain horizontal? *Ans.* 7 ft.

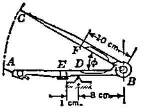


67. A rod AB 15 ft. long weighing 40 lbs. is suspended from ropes at A and B. The tension in the rope at A is 20 lbs. and at B it is 40 lbs. At points C, D, E, and F, spaced so that $AC = CD = DE = EF = FB$, the loads 10, 20, 30, and 40 lbs. are suspended. At what distance from A should a single support be placed to keep the rod horizontal? *Ans.* At the center of the beam.



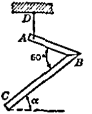
68. Several rectangular plates equal in size and weight are stacked so that each plate overhangs the plate below it. The length of the plates is $2l$. What is the maximum overhang possible for each plate under stable conditions?

Ans. l, $\frac{1}{2}l$, $\frac{1}{3}l$, $\frac{1}{4}l$, $\frac{1}{5}l$, etc.

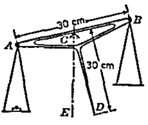


69. A compass with leg AB weighing 16 grams and leg CB weighing 12 grams is balanced on the knife edge D. The center of gravity of AB is at E and that of CB is at F. $BD = 8$ cm., $ED = 1$ cm., and $BF = 10$ cm. To what angle ϕ must the compass be opened so that AB will lie in a horizontal position?

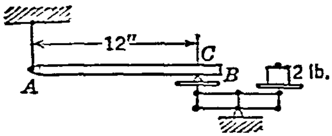
Ans. $\cos\phi = \frac{2}{3}$. (The equilibrium is unstable. It would be stable if the compass were turned 180°.)



70. A bell crank ABC, with the angle between the arms $= 60°$ and $CB = 2AB$, is suspended from point A. Find the angle α between BC and the horizontal. *Ans.* $\tan\alpha = \frac{1}{5}\sqrt{3}$.

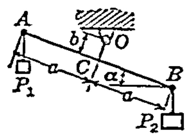


71. A balance beam AB 30 cms. long weighs 300 grams. The pointer CD is 30 cm. long. A difference in weight in the pans of 0.01 gram moves the end of the pointer a distance $ED = 0.3$ cm. How far is the center of gravity of the beam from knife edge C? *Ans.* 0.05 cm.



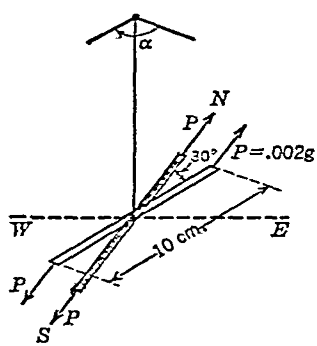
72. A rod AB weighing 3 lbs. is suspended at one end A. Point C, 12″ from A, is supported in the pan of a balance. The scales are levelled by a weight of 2 lbs. How far is the center of gravity of the rod from A?

Ans. 8 in.



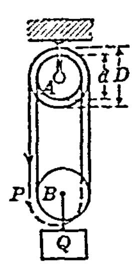
73. Rod OC is connected perpendicularly to the middle of rod AB. This system can rotate about the point O. Each rod weighs $2w$ per unit length. $OC = b$ and $AB = 2a$. A weight P_1 is suspended at A and P_2 is suspended at B. $P_2 > P_1$. What is the angle α between AB and the horizontal at equilibrium?

$$Ans.\quad \tan\alpha = \frac{a}{b} \times \frac{P_2 - P_1}{P_2 + P_1 + w(4a + b)}.$$



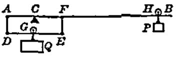
74. A magnetic needle is suspended on a thin wire and placed horizontally in the plane of the magnetic meridian. The horizontal component of the earth's magnetic field is such that opposite forces equal to 0.002 gram act on each of the poles of the needle which are 10 cm. apart. The torsional stiffness of the wire is 0.005 gram cm. per degree twist. Through what angle should the upper end of the suspension be turned to bring the needle to a position making 30° with the plane of the magnetic meridian?

Ans. 32°.



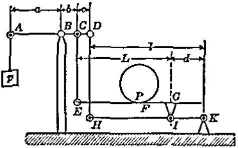
75. A differential chain block consists of two concentric pulleys rigidly attached together, rotating on a fixed axis. The pulleys form two sprockets for an endless chain looped about them in two loops. In one loop a movable pulley B is mounted carrying a load Q. A force P is applied to the proper side of the other loop. The pulley diameters are D and d, $D > d$. Neglecting friction, find the force P necessary to lift Q? What is P for $Q = 1000$ lbs., if $D = 12\frac{1}{2}$ in., and $d = 12$ in.?

Ans. $P = 20$ lbs



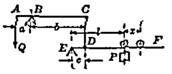
76. A differential lever consists of two parallel bars AB and DE, DE being suspended from AB by two parallel links AD and FE. Bar AB rests on a fulcrum C half way between A and F. A weight Q is suspended on a knife edge at G. It is balanced by a weight P at H. If $AC = 10$ in., $DG = 9.96$ in., $CH = 40$ in., and $Q = 2000$ lbs., what is the weight P? *Ans.* $P = 2$ lbs

77. A scale is constructed from a system of levers as shown in the sketch. All joints shown are freely rotatable A weight P is placed on the scale platform EG at the point F and is balanced by a weight p hung at A. Let $AB = a$, $BC = b$, $CD = c$,

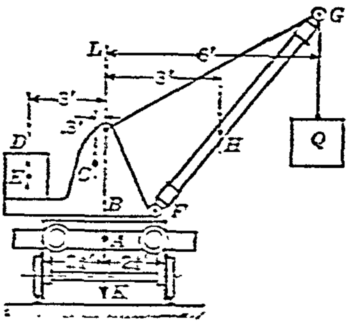


$IK = d$, $HK = l$, and $EG = L$ What should be the relationship between the lengths b, c, d, and l, in order that the balance-weight p will be independent of the position of P on the platform? Under this condition, what is the weight p?

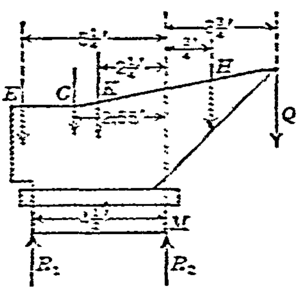
Ans $\dfrac{b + c}{b} = \dfrac{l}{d}$, $p = P\dfrac{b}{a}$



78. A system of two levers ABC and EDF is used to measure a large weight Q suspended at A They are connected by a link CD and are pivoted at B and E. $a = 0.15$ in., $b = 30$ in., and $c = 2$ in A 25-lb balancing weight P can be moved along EDF. The weight Q is balanced by P when it is a distance l from E If Q is increased by 2000 lbs., how much should P be shifted to balance the increased weight? *Ans.* 0.8 in.



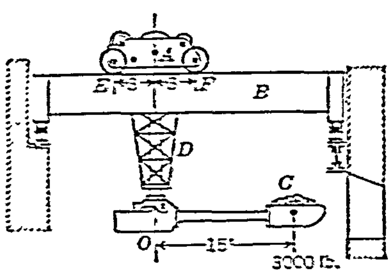
79. A railway crane stands on rails 4.5 ft. apart. The crane truck weighs 6000 lbs. Its center of gravity is on the center line LK. The hoisting gear weighs 2000 lbs.; its center of gravity C is 0.3 ft. from LK. The counterweight D weighs 4000 lbs.; its center of gravity E is 3 ft. from LK. The boom FG weighs 1000 lbs. Its center of gravity H, in a certain position, is 3 ft. from LK. In this boom position, the load line is 6 ft. from LK. Under these conditions, what weight Q will tip over the crane?



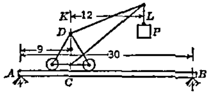
Solution:

The crane will tip over when the reaction R_1 is zero. Taking the moments of all forces about the rail M, the equilibrium of the system yields the equation (§ 16): $5.25 \times 4000 + 2.55 \times 2000 + 2.25 \times 6000 - 4.5 \times R_1 - .75 \times 1000 - 3.75Q = 0$. With $R_1 = 0$, $Q =$ *10,360 lbs.*

80. A loading crane for an open-hearth furnace consists of a trolley A which can run on the rails of a movable bridge B. An inverted column D attached to the trolley carries a scoop C. The center of gravity of the trolley column and empty scoop

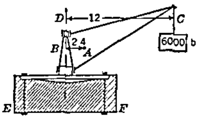


is on the center line of the trolley OA. The wheels of the trolley are 6 ft. apart. A load of 3000 lbs. is placed in the scoop 15 ft. from OA. How much should the trolley column and scoop weigh to keep the trolley from tipping? *Ans.* 12,000 lbs. or more.



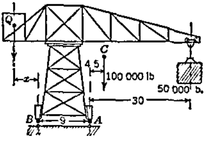
81 The rails of a crane are mounted on a 6000-lb girder AB, 30 ft long The crane weighs 10,000 lbs, and its center of gravity is on the center line CD The overhang of the crane $LK = 12$ ft A load $P = 2000$ lbs is lifted by the crane When $AC = 9$ ft what are the reactions of the supports A and B?

Ans $R_A = 10{,}600$ lbs, $R_B = 7400$ lbs



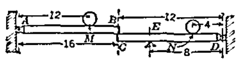
82 A crane weighing 5000 lbs is mounted on a stone foundation Its center of gravity at A is 2 4 ft from the center line of the crane The radius of the crane is 12 ft The foundation has a square base 6 ft on a side and the stone weighs 135 lbs per cu ft The crane lifts a 6000 lb load How deep should the foundation be to keep the whole system from tipping over?

Ans 3 4 ft or more



83 A travelling crane, weighing 100 000 lbs without the counterweight, runs on rails A and B, 9 ft apart Its center of gravity is 4 5 ft to the right of A and the outermost load line is 30 ft to the right of A The lifting capacity of the winch is 50,000 lbs Find the minimum counterweight Q and its maximum distance x to the left of rail B so that the crane is stable under all possible loadings and positions of the carriage

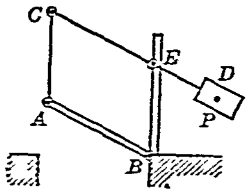
Ans Min $Q = 66{,}700$ lbs Max $x = 20\ 25$ ft



84 A beam AB, 16 ft long and weighing 400 lbs, is hinged at A and rests at B on beam CD which is 12 ft long and weighs 320 lbs CD is hinged at D and supported at E

Two 160-lb. weights are placed at M and N. $AM = 12$ ft., $ED = 8$ ft., $ND = 4$ ft. Find the reactions of the supports.

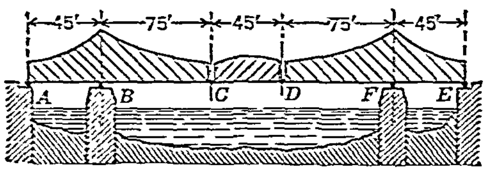
Ans. $R_A = 240$ lbs.; $R_D = 0$; $R_E = 800$ lbs.



85. A drawbridge AB weighing 6000 lbs. is lifted by two levers CD, one on each side. CD is 24 ft. long and weighs 800 lbs. $AB = CE = 15$ ft. The length of the chain $CA = BE$. The center of gravity of the bridge is in the middle of AB. Find the weight of the counter-balance P used on each side of the bridge.

Ans. $P = 2770$ lbs.

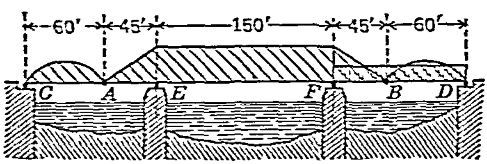
86. A cantilever bridge consists of three parts AC, CD, and DE. The end spans each rest on two supports. The bridge



weighs 4000 lbs. per linear foot. $AC = DE = 120$ ft., $CD = 45$ ft., $AB = EF = 45$ ft. Find the reactions of supports A and B.

Ans. $R_B = 880{,}000$ lbs.; $R_A = 310{,}000$ lbs.

87. A cantilever bridge consists of a main truss AB and two short trusses AC and BD. The main truss weighs 1500 lbs. per linear foot; the short trusses weigh 1000 lbs. per linear foot.

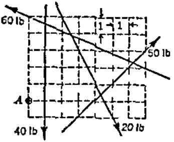


$AC = BD = 60$ ft. $AE = FB = 45$ ft. $EF = 150$ ft. A train weighing 3000 lbs. per linear foot stands on the bridge extending from D to F. What are the reactions of all the supports?

Ans. $R_C = 30{,}000$ lbs.; $R_D = 120{,}000$ lbs.; $R_E = 162{,}750$ lbs.; $R_F = 482{,}250$ lbs.

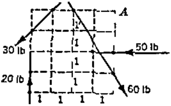
3 General Case of Coplanar Forces.

88 (*a*) Determine the resultant of the four forces shown



(*b*) Determine a single force acting through point A and a couple which together form a system equivalent to the four forces shown

Ans. (*a*) $R = 11.0$ lbs, $\theta_x = 5° 40'$.
(*b*) $F = 11.0$ lbs;
$C = 297$ lb-ft, counterclockwise.

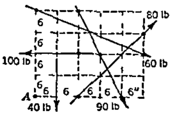


89. (*a*) Determine the resultant of the four forces shown

(*b*) Replace the four forces by an equivalent system consisting of a single force passing through point A and a couple

Ans (*a*) $R = 63.7$ lbs, $\theta_x = 53° 20'$
(*b*) $F = 63.7$ lbs, $C = 16.4$ lb-ft, clockwise

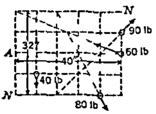
90 (*a*) Determine the resultant of the five forces shown



(*b*) Determine a force system with a single force through A and a couple, equivalent to the system shown

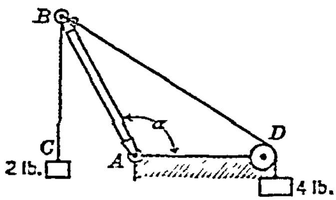
Ans (*a*) $R = 100$ lbs, $\theta_x = 58° 30'$,
(*b*) $F = 100$ lbs,
$C = 1630$ lb-in, clockwise

91. The plate NN is subjected to the tensions of four cables as shown (*a*) *Find the resultant of these forces* (*b*) *Replace* the force system shown by an equivalent system consisting of a single force applied at point A and a couple

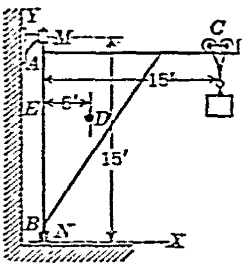


Ans. (*a*) $R = 50$ lbs, $\theta_x = 30° 25'$,
(*b*) $F = 50$ lbs,
$C = 890$ lb-in, counterclockwise.

92. A 2-lb. weight is attached at B to rod AB which is hinged at A. A 4-lb. weight is suspended from a rope attached at B which

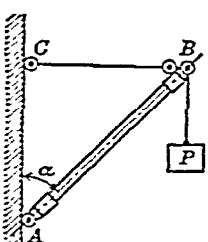


passes over a pulley D. The rod weighs 4 lbs. $AB = AD = 3$ ft. Find the angle α at equilibrium. *Ans.* $\alpha = 120°$.



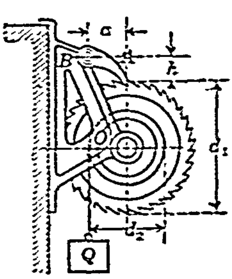
93. A slewing jib-crane ABC weighing 4000 lbs. turns around axis MN. The center of gravity of the crane is at D, 6 ft. from MN. $MN = AC = 15$ ft. A weight of 6000 lbs. is lifted at C. Find the reactions of the bearings M and N.

Ans. at M: $R_x = -7600$ lbs., $R_y = 0$;
at N: $R_x = 7600$ lbs., $R_y = 10{,}000$ lbs.



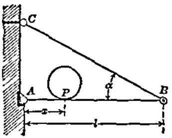
94. A crane consists of a beam AB weighing 200 lbs., hinged at A and held in a position 45° to the vertical by the horizontal rope CB. A load P weighing 400 lbs. is hung at B. Find the tension in the rope CB and the vertical component of the force acting on the hinge A.

Ans. $T_C = 500$ lbs.; $Y_A = 600$ lbs.



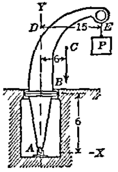
95. A winch is equipped with a ratchet of diameter d_1 rigidly attached to the drum whose diameter is d_2, and a pawl A. A cable wound around the drum carries a load $Q = 100$ lbs., $d_1 = 10.5$ in.; $d_2 = 6$ in.; $h = 1.25$ in.; $a = 3$ in. Find the force acting on the pawl pin B.

$$\textit{Ans.} \quad R = Q\frac{d_2}{d_1}\sqrt{1 + \frac{h^2}{a^2}} = 62 \text{ lbs.}$$



96. A horizontal crane beam of length l is hinged at A and supported at B by the tension rod BC which makes an angle α with the beam. The load P can be applied at any point of the beam. The distance $AP = x$. Find the tension in BC as a function of x.

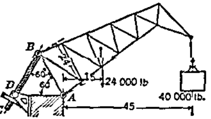
Ans. $T = \dfrac{P\,x}{l \sin \alpha}$.



97. A pit crane weighing 4000 lbs rotates on a step-bearing at A and rests against a smooth cylindrical surface at B. The column AB is 6 ft long. The radius $DE = 15$ ft. The center of gravity of the crane is at C, 6 ft from AY. A weight $P = 8000$ lbs is suspended from E. Find the reactions of the supports A and B.

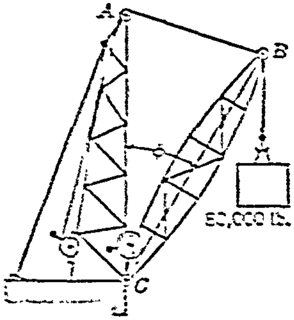
Ans $R_{Ax} = 24{,}000$ lbs , $R_{Ay} = 12{,}000$ lbs , $R_{Bx} = -24{,}000$ lbs., $R_{By} = 0$

98. A crane truss weighing 24,000 lbs is pivoted at A. Its position is adjusted by means of a long bolt BD hinged to the truss at B and passing through a large nut at D. $AB = AD = 24$ ft. When the adjustment is such that ABD is an equilateral triangle, the center of gravity of the truss is on a line



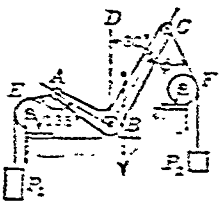
15 ft to the right of A, and the span of the crane is 45 ft. A load of 40,000 lbs is lifted. Find the reactions of the supports and the tension T in the rod.

Ans. $T = 104{,}000$ lbs , $R_{Ax} = 52{,}000$ lbs , $R_{Ay} = 154{,}000$ lbs.



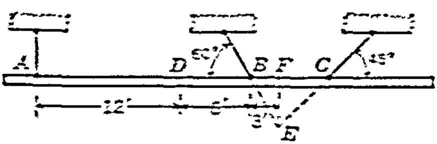
99. A crane consists of a tower AC and a boom BC pivoted at C and held in position by a cable AB. $AC = BC = 45$ ft. A weight of 80,000 lbs. hangs on a chain which passes over a pulley at B and is wound around a drum of a winch near C. The chain extends along the line BC. Neglecting the weight of the boom, find the tension T in the cable AB and the force P compressing the boom as functions of the angle ϕ.

Ans. $T = 160{,}000 \sin (\phi/2)$ lbs.; $P = 80{,}000$ lbs., at any angle ϕ.



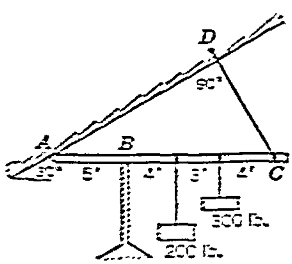
100. A bell crank ABC weighing 16 lbs. is pivoted at B. Its center of gravity is on a line $1.5\sqrt{2}$ in. to the right of line BD. $AB = 4$ in., $BC = 10$ in. BC forms an angle $CBD = 30°$ with the vertical. Ropes attached at A and C pass over pulleys at E and F and carry weights $P_1 = 62$ lbs. and $P_2 = 20$ lbs. Angle $BAE = 135°$. Find the angle $BCF = \phi$ at equilibrium.

Ans. $\phi = 45°$ or $135°$.



101. During the building of a bridge a girder AC weighing 8400 lbs. with its center of gravity at D had to be lifted by three cables placed as shown in the sketch. $AD = 12$ ft., $DB = 6$ ft., $BF = 3$ ft. AC is kept horizontal. Find the tensions in the cables.

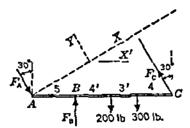
Ans. $T_A = 3600$ lbs.; $T_B = 3515$ lbs.; $T_C = 2485$ lbs.



102. Beam AC, rigged to carry two loads 200 lbs. and 300 lbs. as shown, rests on the top of a wall at B. It is propped up against the roof at A and is supported by cable CD. Determine the forces acting on the bar at A, B, and C, neglecting friction at A and B. (*a*) Solve algebraically. (*b*) Make a graphical construction to solve.

Solution

Bar ABC is in equilibrium under the action of the forces shown Taking X and Y axes as indicated, the equilibrium equations give (§ 21)



$$\Sigma F_x = 0 = F_B \sin 30^\circ - 200 \sin 30^\circ - 300 \sin 30^\circ,$$

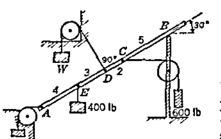
$$F_B = 500 \text{ lbs},$$

$$\Sigma M_A = 500 \times 5 - 200 \times 9 - 300 \times 12 + 16F_C \times 866 = 0,$$

$$F_C = \frac{2900}{16 \times 866} = 209 \text{ lbs},$$

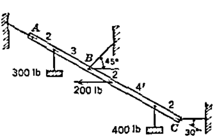
$$\Sigma F_x = F_A \sin 30^\circ - 209 \sin 30^\circ = 0,$$

$$F_A = 209 \text{ lbs}$$



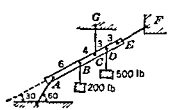
103. A floating beam AB rests against a wall B, supports the loads shown at C and E and is held inclined by cords at A and D Neglecting the frictional effect at B, find the forces acting on the beam at A, D, and B (*a*) Solve algebraically. (*b*) Solve graphically

Ans. $T_A = 369$ lbs, $T_D = 340\ 4$ lbs, $F_B = 121\ 6$ lbs



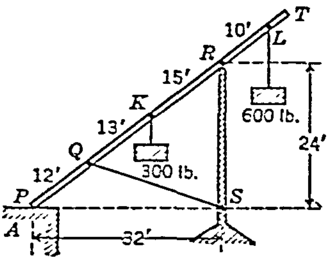
104. Ladder AC is rigged up on three rods at A, B, and C, as shown It carries two weights of 300 lbs and 400 lbs and a horizontal cable exerts a pull of 200 lbs Find the forces exerted by the tie rods on the beam at A, B, and C (*a*) Solve algebraically. (*b*) Solve graphically

Ans $T_C = 374$ lbs ; $T_B = 537$ lbs, $T_A = 639$ lbs



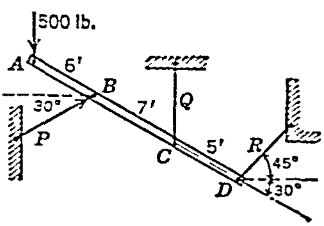
105. The bar AE shown supports two loads and is held in the inclined position by the three cords at A, C, and E (*a*) Determine the tensions in the three cords using algebraic methods (*b*) Find the tensions in the cords using a graphical solution

Ans. $T_A = 121$ lbs, $T_C = 770$ lbs, $T_E = 70$ lbs



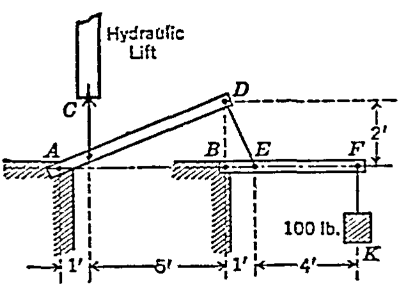
106. Roof beam PT rests on walls A and RS as shown, and is held in position by tie rod QS. The beam carries roof loads at K and L as shown. Find the reactions at P and R, and the tension in QS, neglecting frictional effects. (*a*) Solve algebraically. (*b*) Solve graphically.

Ans. $F_P = 362$ lbs.; $F_R = 887$ lbs.; $T_Q = 559$ lbs.



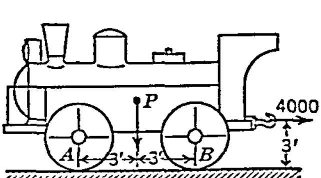
107. Beam AD is held in position by three rods P, Q, and R. The beam carries a load of 500 lbs. at A. Find the amount and kind of force in each of the bars.

Ans. $F_P = 470$ lbs. Compr.;
$F_Q = 672$ lbs. Tens.;
$F_R = 576$ lbs. Compr.



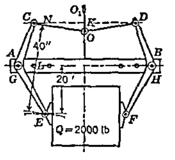
108. A hydraulic lift C is used to operate gate K through beams AD and BF, tied together by cable DE. Neglecting friction, find the force of lift C, the reactions at A and B and the tension in DE for equilibrium in the position shown.

Ans. $T_{DE} = 559$ lbs.;
$F_C = 3500$ lbs.

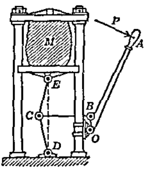


109. A two axle locomotive weighing 40,000 lbs. exerts a drawbar pull of 4000 lbs. With the dimensions given in the sketch, find the axle loadings at A and B.

Ans. $R_A = 18{,}000$ lbs.;
$R_B = 22{,}000$ lbs.

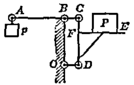


110. The chain OO_1 of a lifting tong is connected to two bell cranks CAE and DBF by links OC and OD each 24 inches long. The bell cranks pivot about pins A and B in the cross bar GH. The pivot blocks E and F hold a load $Q = 2000$ lbs. by means of friction. $EL = 20$ inches, $EN = 40$ inches, $OK = 4$ inches. Neglecting the weight of the mechanism, find the tension in the cross bar GH. *Ans.* $T = 12{,}000$ lbs.



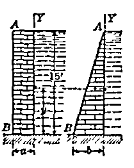
111. The lever OA of a press is 3 ft. long and is pivoted at O. A workman exerts a pull $P = 40$ lbs. at A in a direction perpendicular to OA. At a certain time the link CB is perpendicular to OB and bisects the angle ECD. Angle $CED = 11°\ 20'$; $OB = 0.3$ ft. What is the compressive force acting on M at that time?

Ans. $P_M = 1000$ lbs.



112. A lever of the first kind ABC, pivoted at B, carries a platform CDE hinged to ABC at C and to OD at D. OD is pivoted at O and is equal and parallel to BC. CD is vertical and $BC = 0.1AB$. A weight $P = 200$ lbs. is put on the platform. What weight p is necessary at A to balance the system? *Ans.* $p = 20$ lbs.

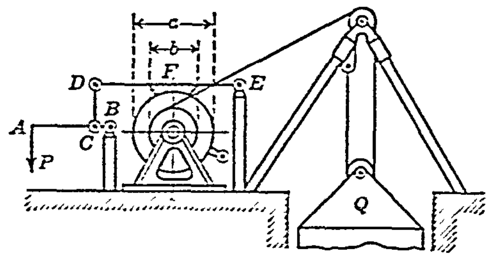
113. The water pressure against any point on the vertical surface of a dam is proportional to the depth of that point below the water level. The surface of the water is on the same level as the top of the dam. The height of the dam is 15 ft.; water weighs 62.3 lbs. per cu. ft. and stone weighs 137 lbs. per cu. ft. The dam by its own weight and dimensions should resist being tipped over the edge B by the moment of the water thrust. For safety the resisting moment should be twice as great as the water thrust moment



Find the necessary thickness a, if the cross section is rectangular, and the base b, if it is triangular in shape.

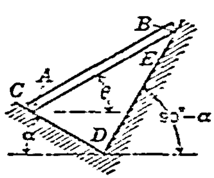
Ans. $a = 8.25$ ft.; $b = 10.12$ ft.

114. A drum 10 inches in diameter rigidly attached to a concentric wooden brake wheel 25 inches in diameter is used to lower a load Q into a shaft as shown in the sketch. The braking is accomplished by pressing down at A. The lever AB is connected to the brake arm DE by the chain CD. $ED = 60$ inches; $FE = 30$



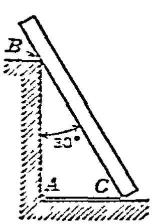
inches; $AB = 50$ inches; $BC = 5$ inches. The brake shoe F is made of cast iron. The coefficient of friction between cast iron and wood is $f = 0.4$. Neglecting the dimensions of the shoe, find the force P at A necessary to balance a weight $Q = 1600$ lbs.

Ans. $P = 40$ lbs.



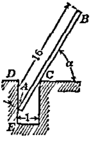
115. A board AB weighing W lbs. rests on two smooth boards CD and DE which are perpendicular to each other. CD makes an angle α with the horizontal. At equilibrium, find the angle θ between AB and the horizontal and the reactions at A and B.

Ans. $\theta = 90° - 2\alpha$, ($\alpha \leqq 45°$); $R_A = W \cos \alpha$; $R_B = W \sin \alpha$.

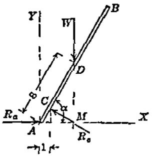


116. A log 12 ft. long weighing 120 lbs. rests with one end on a smooth floor and with the point B against the upper edge of a wall 9 ft. high. It makes an angle of 30° with the vertical. The log is held in this position by a rope AC stretched on the floor. Neglecting the effect of friction, find the tension T in the rope and the reactions at B and C.

Ans. $T = 30$ lbs.; $R_B = 34.6$ lbs.; $R_C = 102.7$ lbs.



117. A rod AB 16 ft long weighing 40 lbs rests against a vertical wall DE and leans against the corner C of another wall 1 ft from DE. Find the angle α between the rod and the horizontal at equilibrium. What are the reactions at A and C? Neglect friction.



Solution

The reaction R_a at A is normal to the wall, the reaction R_c at C is normal to AB. Considering the equilibrium of the three forces W, R_a, R_c (§ 21)

$$\Sigma F_x = R_a - R_c \sin\alpha = 0 \qquad R_a = R_c \sin\alpha$$

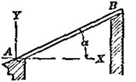
$$\Sigma F_y = R_c \cos\alpha - W = 0 \qquad R_c = \frac{W}{\cos\alpha},$$

$$\Sigma M_A = R_c \frac{1}{\cos\alpha} - W \times 8\cos\alpha = 0$$

Solving we find $\cos^3\alpha = \frac{1}{8}$, $\cos\alpha = \frac{1}{2}$, $\alpha = 60°$, $R_c = 2W = 80$ *lbs*

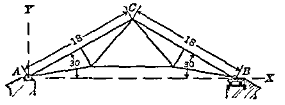
$$R_a = W\sqrt{3} = 69.2 \text{ lbs}$$

NOTE. The fact that the three forces must be concurrent (§ 10a) intersecting at M, could be used for solving this problem.



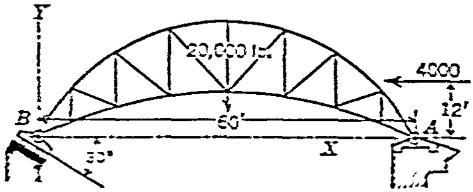
118. A rafter AB rests on a wall at A and on a smooth support at B. The rafter forms an angle $\alpha = \tan^{-1} 0.5$ with the horizontal. The rafter carries a vertical load of 1800 lbs at the middle. Find the reactions at A and B.

Ans. $R_B = 804$ lbs, $F_{Ax} = 360$ lbs, $F_{Ay} = 1080$ lbs



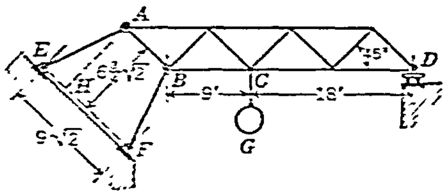
119. A symmetrical roof truss ABC weighing 20 000 lbs is hinged at one point A. The other end B rests on rollers which roll on a horizontal plate. $AC = BC = 18$ ft, angle $CAB = 30°$. A uniformly distributed wind load of 1600 lbs acts normally on AC. Find the reactions at A and B.

Ans. $R_B = 10{,}460$ lbs, $F_{Ax} = 800$ lbs, $F_{Ay} = 10{,}920$ lbs

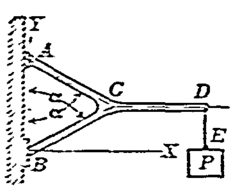


120. An arched truss AB 60 ft. long, weighing 20,000 lbs., is hinged at A. At B it rests on rollers which can roll on a plate inclined 30° to the horizontal. A wind blowing parallel to AB exerts a force of 4000 lbs. on the structure; its line of action is 12 ft. above AB. Find the reactions of the supports.

Ans. $F_{Ax} = 2240$ lbs.; $F_{Ay} = 9200$ lbs.; $R_B = 12{,}480$ lbs.

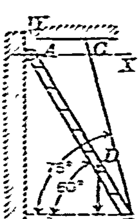


121. A truss $ABCD$ rests on rollers at D. It is supported at A and B by struts EA and BF hinged at E and F. The braces of the truss and line EF are at 45° to the horizontal. Girder BC is 9 ft. long. $AE = BF$. $EF = 9\sqrt{2}$ ft. $AH = 6\frac{3}{4}\sqrt{2}$ ft. The weight of the truss, supporting struts and load equals 15,000 lbs. and may be considered as acting along line CG. Find the reaction R at D. *Ans.* $R_D = 3000$ lbs.



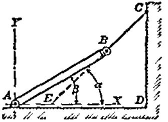
122. A hanger consists of three equal legs AC, BC, and CD rigidly attached together at C. Each leg weighs p lbs. The hanger is hinged at A and rests against a smooth vertical wall at B. A weight P is suspended from D. Find the reactions at A and B.

$$Ans.\quad R_{Bx} = \frac{2P + p + 2(P + 2p)\sin\alpha}{4\cos\alpha} = -R_{Ax};\ R_{Ay} = P + 3p.$$



123. A fire escape ladder 14 ft. long, weighing 600 lbs., is hinged at A and suspended at B on a chain CB. The ladder forms an angle of 60° and the chain forms one of 75° with the horizontal. A man weighing 200 lbs. stands at D, $4\frac{2}{3}$ ft. from B. Find the tension in the chain and the reaction at A.

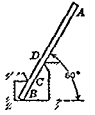
Ans. $T = 837$ lbs.; $F_{Ax} = 217$ lbs.;
$F_{Ay} = 8.5$ lbs. (downward).



124 A rod AB weighing W lbs., and $2l$ ft. long, is hinged at A to the floor AD. The other end is tied to the wall CD by the rope BC. Angle $CED = \alpha$ and $BAD = \beta$. Find the reaction of the hinge A and the tension in the rope.

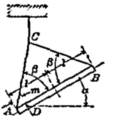
Ans. $T = \dfrac{W \cos \beta}{2 \sin (\alpha - \beta)}$;

$$F_{Ax} = \frac{W \cos \beta \cos \alpha}{2 \sin (\alpha - \beta)} \text{ (to left)}; \quad F_{Ay} = W\left(1 - \frac{\cos \beta \sin \alpha}{2 \sin (\alpha - \beta)}\right).$$



125. A beam AB, 9 ft. long, weighing 40 lbs., rests on a floor at B, forming an angle of 60° with the horizontal. The beam is supported at C and D. $CB = 1.5$ ft.; $BD = 3$ ft. Find the reactions at B, C, and D.

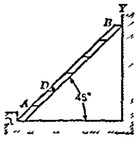
Ans. $R_C = R_D = 60$ lbs.; $R_B = 40$ lbs.



126. A board AB $2l$ ft. long, weighing w lbs., hangs on two ropes AC and BC equal in length The angles between the ropes and the board equal β. A man weighing W lbs. stands at D, m ft. from A. In the position of equilibrium, what is the angle α between the board and the horizontal? What are the tensions T_A and T_B in the ropes?

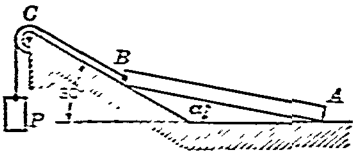
Ans. $\tan \alpha = \dfrac{(l - m)W}{(w + W)l \tan \beta}$; $T_A = (W + w)\dfrac{\cos (\beta - \alpha)}{\sin 2\beta}$;

$$T_B = (W + w)\frac{\cos (\beta + \alpha)}{\sin 2\beta}.$$



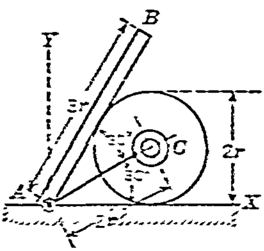
127. A ladder AB weighing 40 lbs leans against a smooth wall and is braced against a step A. It forms an angle of 45° with the horizontal. A man weighing 120 lbs. stands at D, ⅓ the distance up the ladder. Find the reactions of the wall and of the step A.

Ans. $R_B = 60$ lbs.; $F_{Ax} = 60$ lbs. (to right); $F_{Ay} = 160$ lbs.



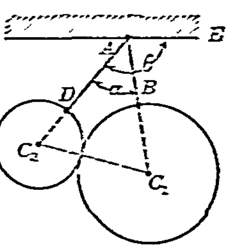
128. A log AB weighing 200 lbs. rests with one end A on a smooth horizontal floor and with the other end on a smooth plane inclined at an angle of 30°. B is attached to a rope passing over a pulley C and loaded with a weight P. The rope between B and C is parallel to the inclined plane. Find the weight P and the reactions at A and B.

Ans. $R_A = 100$ lbs.; $R_B = 86.6$ lbs.; $P = 50$ lbs.



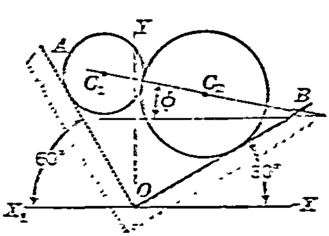
129. A bar AB of length $3r$, weighing 32 lbs., is hinged at A and rests on a smooth cylinder of radius r. The cylinder lies on a smooth horizontal plane and is held in position by a rope AC, $2r$ long. Find the tension T in the rope and the reaction on bar AB at A.

Ans. $F_{Ax} = 12$ lbs.; $F_{Ay} = 25.1$ lbs.;
$T = N = 13.8$ lbs.



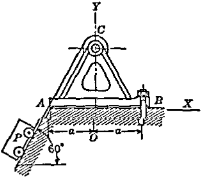
130. Two balls C_1 and C_2, the radii of which are R_1 and R_2, are suspended from A by ropes AB and AD. $AB = l_1$; $AD = l_2$; $l_1 + R_1 = l_2 + R_2$. The balls weigh W_1 and W_2. Find the angle θ between the rope AD and the horizontal, the tensions T_1 and T_2 in the ropes, and the reaction N between the balls.

Ans. $$\tan\theta = \frac{W_2 + W_1\cos\alpha}{W_1\sin\alpha};\quad T_1 = W_1\frac{\sin(\theta-\alpha/2)}{\cos\alpha/2};$$
$$N = -W_2\frac{\cos\theta}{\cos\alpha/2};\quad T_2 = W_2\frac{\sin(\theta-\alpha/2)}{\cos\alpha/2}.$$



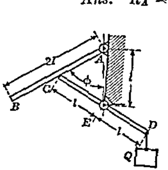
131. Two balls touching each other rest on inclined planes OA and OB. One ball with its center at C_1 weighs 20 lbs. and has a radius of 1 inch; the other with its center at C_2 weighs 60 lbs. and has a radius of 2 inches. Angle $AOX_1 = 60°$ and angle $BOX = 30°$. Find the angle ϕ between line C_1C_2 and the horizontal,

the reactions N_1 and N_2 of the planes, and the force N between the balls. *Ans.* $\phi = 0; N_1 = 40$ lbs.; $N_2 = 69.2$ lbs.; $N = 34.6$ lbs.



132. A weight $P = 960$ lbs. is held on a smooth plane by means of a rope wound around the drum of a winch ABC. The plane is inclined at 60° to the horizontal. The rope is parallel to the plane. The winch weighs $Q = 480$ lbs., rests on a smooth floor at A and is bolted to the floor at B; its center of gravity is on line CO. Find the reactions of the supports, neglecting the distance between the rope and the plane.

Ans. $R_A = 958$ lbs.; $R_{Bx} = 415$ lbs.; $R_{By} = 240$ lbs.



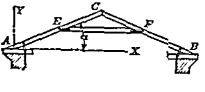
133. A rod AB weighing W lbs., $2l$ ft. long, is hinged at A. It rests on a rod CD, $2l$ ft. long, which is hinged at its middle point E. A and E are l ft. apart on the same vertical line. A weight $Q = 2W$ is suspended from D. Neglecting the effect of friction, find the angle ϕ in the position of equilibrium.

Ans. $\phi_1 = 0; \phi_2 = \cos^{-1} \frac{1}{8}$.

134. A roof truss consists of two 15-ft. timbers joined at C and held together by a horizontal tie beam AB at their lower ends.

The roof is inclined to the horizontal at an angle $\alpha = \tan^{-1} 0.5$. A load of 1800 lbs. is applied at the middle of each timber. Find the forces acting at C and at A.

Ans. $F_C = 1800$ lbs.; $F_{Ax} = 1800$ lbs.; $F_{Ay} = 1800$ lbs.

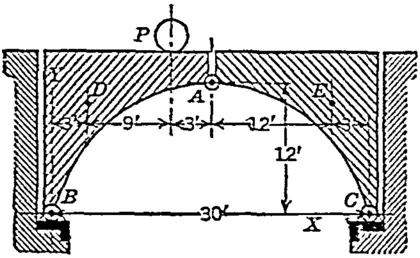


$\alpha = \tan^{-1} 0.5$. $AC = 3CE$.

135. A roof truss consists of two 12-ft. timbers AC and BC resting on two walls at A and B and held together by a tie beam EF. The angle of the roof is $\alpha = \tan^{-1} 0.5$, $AC = 3CE$. A load of 1600 lbs. is applied to the

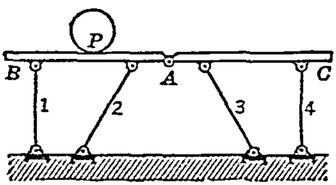
middle of each beam AC and BC. Neglecting the effects of friction, find the reaction of the wall A and the tension T in EF.

Ans. $R_A = 1600$ lbs.; $T = 4800$ lbs.

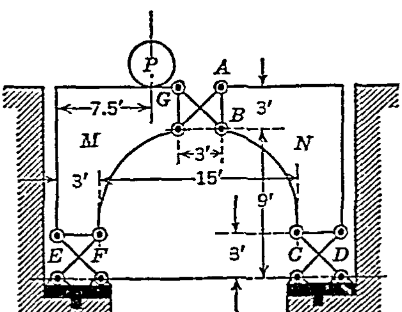


136. A bridge consists of two parts hinged together at A and hinged to the piers at B and C. Each part weighs 8000 lbs., the centers of gravity being at D and E. A load $P = 4000$ lbs. is on the bridge. Dimensions are given in the sketch. Find the force at A and the reactions at B and C.

Ans. $F_{Ax} = 4000$ lbs.; $F_{Bx} = 4000$ lbs.; $F_{Cx} = -4000$ lbs.;
$F_{Ay} = 1600$ lbs.; $F_{By} = 10{,}400$ lbs.; $F_{Cy} = 9600$ lbs.



137. Two horizontal beams are hinged together at A and are supported by four hinged struts 1, 2, 3, and 4. A load P rests on the beam AB. Find the forces in the struts graphically; scale the dimensions from the sketch.

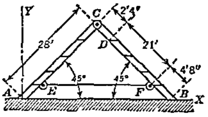


138. A bridge consists of two identical parts M and N connected together and to the piers by means of six equal hinged struts inclined at 45° to the horizontal. A load P is placed at G. Dimensions are given in the sketch. Find the forces in the struts caused by P.

Ans. $F_A = 0$; $F_C = 0$;

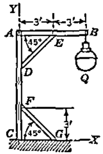
$$F_B = \frac{P}{3}\sqrt{2};\ F_D = \frac{P}{3}\sqrt{2};\ F_E = \frac{P}{2}\sqrt{2};\ F_F = \frac{P}{6}\sqrt{2}.$$

139. A step-ladder consisting of two parts AC and BC hinged at C and held together by a rope EF stands on a smooth floor.



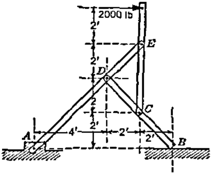
Each part is 28 ft. long and weighs 40 lbs. A man weighing 160 lbs. stands at D. CD = 2 ft. 4 in. and $EA = FB$ = 4 ft. 8 in. Find the reactions of the floor and of the hinge and the tension T in the rope.

Ans. R_A = 113.3 lbs.; F_{Cx} = 112.1 lbs.; T = 112.1 lbs.;
R_B = 126.7 lbs.; F_{Cy} = 73.3 lbs.

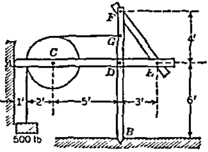


140. *A horizontal arm 6 ft. long is attached* to a vertical post at A and is supported by the brace DE. The post AC is held up by a brace FG. The braces DE and FG are inclined at 45° to the horizontal. $AE = CF$ = 3 ft. A load Q = 1000 lbs. is suspended from B. Find the forces S_E and S_F in the braces DE and FG and the reaction of the ground at the foot C.

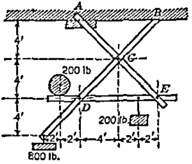
Ans. S_E = 2830 lbs.; F_{Cx} = 2000 lbs.;
S_F = 2830 lbs.; F_{Cy} = 1000 lbs. (downward).



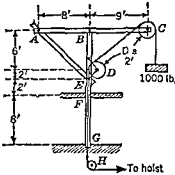
141. The framework shown in the sketch is pin-connected at the points D, C, and E. It is hinged at A and rests on a smooth floor at B. Determine all the forces acting on each bar.



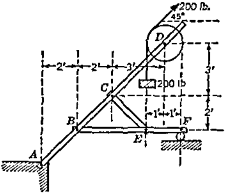
142. The pin-connected framework, shown in the sketch, *is supported in a socket at B* and leans against a smooth vertical wall at A. A load of 500 lbs. is suspended from a cord attached to the framework at G and passing over a pulley at C. Determine the reactions at A and B and all forces acting on the beam $ACDE$.



144. The pin-connected framework shown in the sketch is hinged at A and rests against a smooth ceiling at B. The bars are pinned together at C, D, and E and carry the three loads shown. Determine the reactions at A and B and all forces acting on the bar BCD.

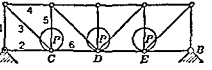


145. A crane consists of three bars pinned together at A, B, and E, as shown. It rests in a socket at G and is supported laterally at F. The crane carries a load of 1000 lbs., hanging on a wire rope passing over pulleys C, D, and H. Determine all the forces which act on each member of the crane.



146. The pinned frame shown is hinged at A and rests on a smooth roller at F. Beam AD carries a pulley D of 2 ft. diameter. A rope carrying a load of 200 lbs. passes over the pulley as shown. Determine the forces acting on each bar.

4. Trusses and Cables.

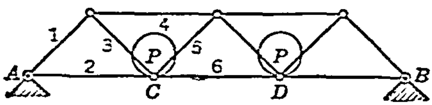


147. A bridge truss as shown in the sketch carries three equal loads $P = 20{,}000$ lbs. at the points C, D, and E. The inclined web members form 45° angles with the horizontal. Find

analytically the forces in members 1, 2, 3, 4, 5, and 6 caused by this loading.

Ans. $S_1 = -30{,}000$ lbs. (compression); $S_2 = 0$;
$S_3 = +42{,}500$ lbs. (tension); $S_4 = -30{,}000$ lbs.;
$S_5 = -10{,}000$ lbs.; $S_6 = +30{,}000$ lbs.

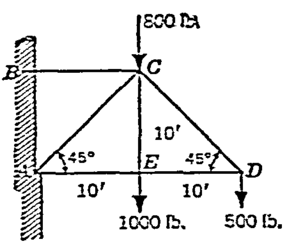
148. A bridge truss as shown in the sketch carries two equal loads $P = 20{,}000$ lbs. at the points C and D. The inclined web



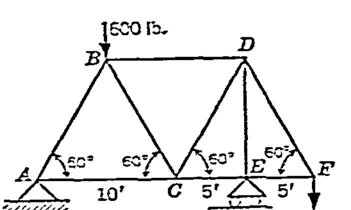
members form 45° angles with the horizontal. Find analytically the forces in members 1, 2, 3, 4, 5, and 6 due to the loading.

Ans. $S_1 = -28{,}200$ lbs.; $S_2 = +20{,}000$ lbs.; $S_3 = +28{,}200$ lbs.;
$S_4 = -40{,}000$ lbs.; $S_5 = 0$; $S_6 = +40{,}000$ lbs.

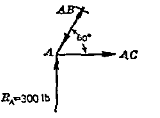
149. This cantilever truss bears the three loads shown and is supported at A and B. Determine algebraically the forces in all the members of the truss.



Ans. $S_{BC} = +2800$ lbs.; $S_{CD} = +707$ lbs.;
$S_{AE} = S_{ED} = -500$ lbs.; $S_{CE} = +1000$ lbs.;
$S_{AC} = -3255$ lbs.



150. This 60° truss carries two loads and rests on two supports. (*a*) Determine algebraically the forces in all the members of the truss. (*b*) Construct a force diagram, and determine from it the forces in the members.

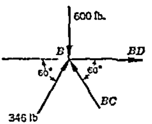


Solution

(*a*) *From a free body diagram of the entire truss, the reactions are found to be* (§ 21)

$$R_A = 300 \text{ lbs}, \qquad R_E = 600 \text{ lbs}$$

Using the free body diagram for each joint of the truss we see that the forces in all the members are determined as follows (§ 10)



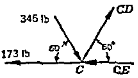
Joint A (Assume AB in tension)

$$\Sigma F_y = 0 = AB \sin 60° + 300,$$

$$AB = -\frac{300}{0.866} = -346 \text{ lbs compression.}$$

$$\Sigma F_x = AC - 346 \cos 60° = 0,$$

$$AC = 346 \times \tfrac{1}{2} = 173 \text{ lbs tension}$$



Joint B

$$\Sigma F_y = 346 \sin 60° + BC \sin 60° - 600 = 0,$$

$$BC = 346 \text{ lbs compression}$$

$$\Sigma F_x = 0 = 346 \cos 60° - 346 \cos 60° + BD = 0$$

$$BD = 0$$

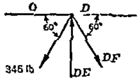
Joint C

$$\Sigma F_y = CD \sin 60° - 346 \sin 60° = 0,$$

$$CD = 346 \text{ lbs tension}$$

$$\Sigma F_x = -173 + 346 \cos 60° + 346 \cos 60° - CE = 0$$

$$CE = 173 \text{ lbs compression}$$



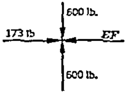
Joint D

$$\Sigma F_x = 0 = DF \cos 60° - 346 \cos 60° = 0$$

$$DF = 346 \text{ lbs tension}$$

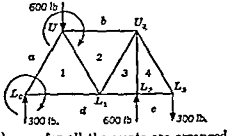
$$\Sigma F_y = DE - 346 \sin 60° \times 2 = 0,$$

$$DE = 600 \text{ lbs compression.}$$

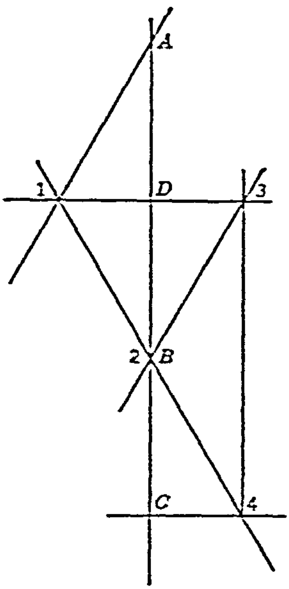


Joint E

$$EF = 173 \text{ lbs compression}$$

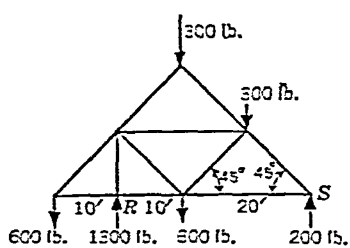


(*b*) The force polygons for all the joints are arranged systematically to give the forces in all members as follows (§ 18)

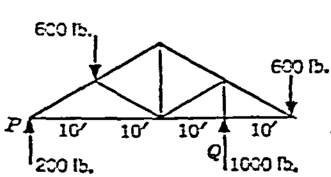


A clockwise stress diagram is constructed. The following "clockwise" notation for the external forces is used: the 600-lb. force at U_1 is ab, the 300-lb. force at L_3 is bc, the 600-lb. force at L_2 is cd, etc. The notation for the force exerted on a joint by any member is determined by reading the letters or numbers around the joint in a clockwise order. The force that member U_1L_1 exerts on joint U_1 is 2–1, and the force this same member exerts on joint L_1 is 1–2. The equilibrium polygon for the external forces is first constructed as $ABCDA$. Then, constructing the force polygon for joint L_0, DA is known, the direction of A–1 is parallel to a–1 and passes through A, and 1–D is parallel to 1–d and passes through D. The intersection of A–1 and 1–D gives point 1. The force exerted on joint L_0 by member L_0U_1 is A–1 in magnitude and direction, and the force exerted by L_0L_1 on joint L_0 is 1–D in magnitude and direction. By next drawing the force polygon for joint U_1, point 2 is located and the forces exerted by members U_1U_2 and U_1L_1 on joint U_1 determined. Continuing in this way, the closed diagram representing the forces in all members of the truss is obtained.

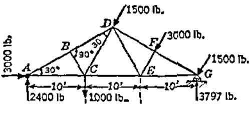
151. The 45° truss in the sketch is supported at R and S. It carries four loads as shown. (*a*) Using algebraic methods, determine the forces in all members of the truss. (*b*) Draw a



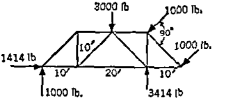
force diagram and determine the amount and kind of force in each member of the truss.



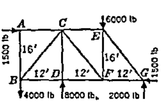
152. A 30° truss carries two 600-lb. loads; the reactions at P and Q are as indicated. (*a*) Determine algebraically the force in each member. (*b*) Draw a force diagram.



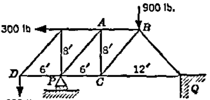
153. A roof truss is supported by a pin at A and a roller at G; it carries four loads as shown. (a) Determine algebraically the forces in all the members of the truss. (b) Draw a force diagram.



154. The sketch gives the loads on a truss and the reactions of the supports. (a) Determine algebraically the forces in all the members. (b) Draw a force diagram.

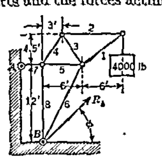


155. The loads and reactions on a truss are shown in the sketch. (a) Determine algebraically the forces in members CE, CF, and DF. (b) Determine the forces in members CE, CF, and DF from a force diagram for the truss.



156. The truss shown in the sketch is supported by a pin at Q and a vertical support at P. It is loaded as shown in the sketch. (a) Find algebraically the forces in members AC, AB, and PC. (b) Find the forces in members AC, PC, and AB from a force diagram for the truss.

157. The crane shown in the sketch lifts 4000 lbs. Find the reactions of the supports and the forces acting in all the members.

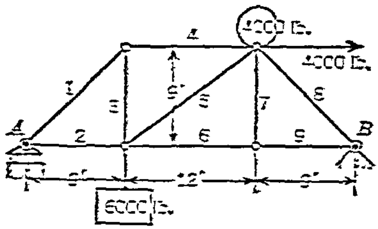


Ans. $R_a = 4000$ lbs.; $R_b = 5600$ lbs.; $\phi = 45°$.

The forces in the members are:

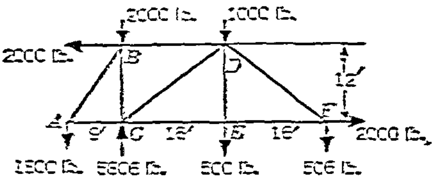
Member No.	1	2	3	4	5
Force, lbs.	−6670	+5330	−4800	+4800	+1333
Member No.	6	7	8		
Force, lbs.	−8950	+4000	+4000		

158. Find the reactions of the supports and the forces in the members of the bridge truss shown in the sketch with forces acting as indicated.

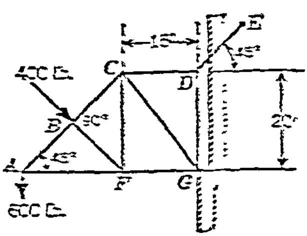


Ans. $R_A = 4200$ lbs.; $R_B = 7040$ lbs.

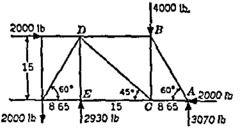
Member No.	1	2	3	4	5
Force, lbs.	−5940	+4200	+4200	−4200	+3000
Member No.	6	7	8	9	
Force, lbs.	+1800	0	−8210	+1800	



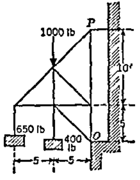
159. The truss shown in the sketch is supported by a pin at F and a roller at C. The loads and reactions are as shown. (*a*) Find algebraically the forces in the truss. (*b*) Construct a force diagram for the truss.



160. A cantilever truss carries two loads and is supported at G and E as shown in the figure. Determine algebraically the forces in each member of the truss.

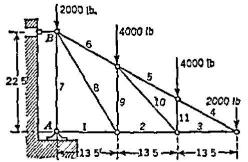


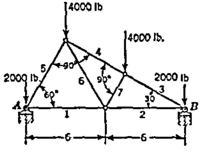
161. The loads and reactions on a truss are given in the sketch (*a*) Find algebraically the forces in all the members of the truss (*b*) Find the forces in all members of the truss from a force diagram



162. A cantilever truss carries three loads and is supported at *P* and *O* as shown (*a*) Determine the amount and kind of force in each member of the truss, using algebraic methods (*b*) Draw a force diagram for the truss

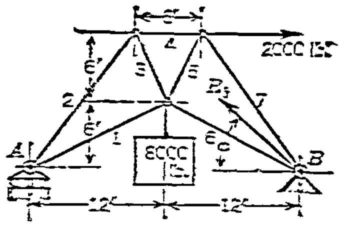
163. Find graphically the forces in the members of the shed roof truss shown in the sketch with the forces acting upon it as indicated.





164 Find graphically the forces in the members of the roof truss shown in the sketch with the forces acting upon it as indicated

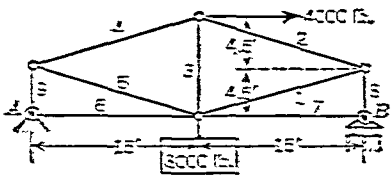
165. The structure shown in the sketch is loaded as indicated. Find the reactions of the supports and the forces acting on the members.



Ans. $R_a = 3000$ lbs.; $R_b = 5600$ lbs.; ($\phi = 68°$).

Member No.	1	2	3	4	5
Force, lbs.	+4020	−6000	+5360	−6000	+7150

Member No.	6	7
Force, lbs.	+3130	−8000



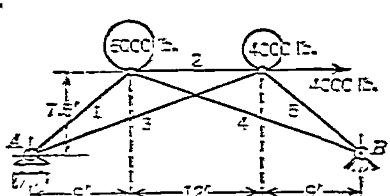
166. Find the reactions of the supports and the forces acting in all the members of the structure shown in the sketch. The loads are as indicated.

Ans. $R_A = 4800$ lbs.; $R_B = 5200$ lbs.

Member No.	1	2	3	4	5
Force, lbs.	+8670	−8670	+4000	−4670	+4670

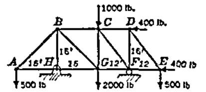
Member No.	6	7	8	9
Force, lbs.	+4000	0	−5200	−2800

167. Find the reactions of the supports and the forces in the members of the structure shown in the sketch. The forces acting are indicated. Members 3 and 4 are not joined at the point of their intersection.

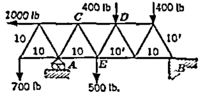


Ans. $R_A = 4400$ lbs.; $R_B = 6800$ lbs.

Member No.	1	2	3	4	5
Force, lbs.	−12,000	+13,990	+9800	+5050	−11,400

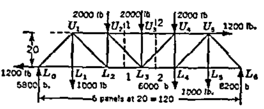


168. The truss shown in the sketch is supported vertically at F and horizontally and vertically at H. (a) Determine algebraically the forces in members BC, BG, and HG. (b) Determine the forces in BC, BG, and HG from a force diagram for the truss

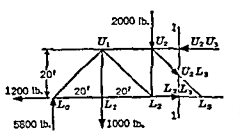


169. This truss bears five loads as shown It is supported on a roller at A and by a pin at B. (a) Find algebraically the forces in members CD, CE, and AE. (b) Draw a force diagram and determine from it the forces in members CD, CE, and EA.

170. For the truss shown, use the method of sections to determine the forces in U_2U_3, U_2L_3, and L_3L_4



Solution·



The forces in members U_2U_3 and U_2L_3 are determined by considering the portion of the structure to the right or left of section 1–1 (§ 21) The forces in the members cut by the section become external forces in the free body diagram for either portion of the structure The forces acting on the portion of the truss shown are in equilibrium

$$\Sigma M_{L_3} = 0 = -5800 \times 60 + 1000 \times 40 + 2000 \times 20 + 20U_2U_3 = 0,$$

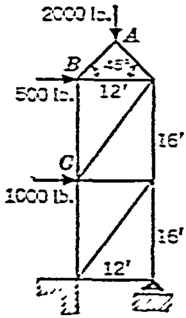
$$U_2U_3 = 13{,}400 \text{ lbs compression},$$

$$\Sigma F_y = 5800 - 2000 - 1000 - U_2L_3 \times 0.707 = 0,$$

$$U_2L_3 = 3960 \text{ lbs tension}$$

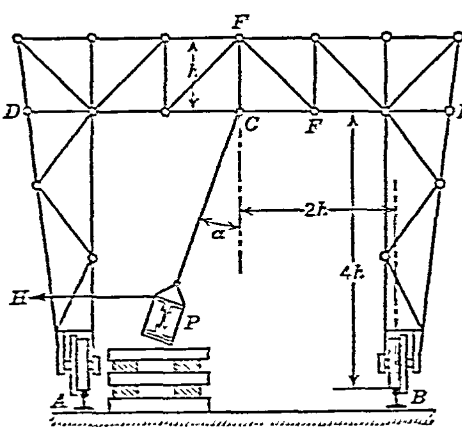
In a similar way, by taking the portion of the structure to the right (or left) of section 2–2, the force in L_3L_4 is obtained.

$$\Sigma M_{U_4} = -1000 \times 20 + 8200 \times 40 - 20L_3L_4 = 0,$$
$$L_3L_4 = 15{,}400 \text{ lbs. tension.}$$



171. The tower shown is subjected to a 2000-lb. vertical load at A and horizontal wind loads at B and C. Determine the forces in each member of the truss.

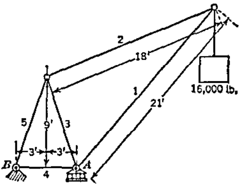
172. During the construction of a bridge a temporary wooden crane was built. It was mounted on wheels which rolled on tracks A and B. A block and tackle was suspended from the midpoint C. A load P of 12,000 lbs. was lifted from a stack to one side of the center; the angle of the chain at the instant of lifting was 20° to the vertical. To prevent the load from



swinging a lateral pull was exerted on it through the rope HP. If the entire horizontal thrust were taken by the rail B, what was the force S_1 in the horizontal member CF at the instant the load was lifted? Compare S_1 to the force S_2 in CF in the case when $\alpha = 0$, with the same load being lifted.

Ans. $S_1 = 25{,}130$ lbs.; $S_2 = 12{,}000$ lbs.

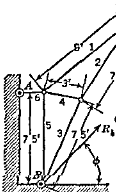
173. The crane shown in the sketch lifts a load of 16,000 lbs. Find the *reactions* of the supports at A and B and the forces in the members.



Ans. The reaction R_B is vertical. $R_A = 52{,}200$ lbs.; $R_B = 36{,}200$ lbs. (downward). The forces in the members are:

Member No.	1	2	3	4	5
Force, lbs.	−32,700	+22,700	−28,700	−12,000	+38,100

174. The crane shown in the sketch lifts a load of 6000 lbs. Find the reaction of the supports and the forces in all the members.

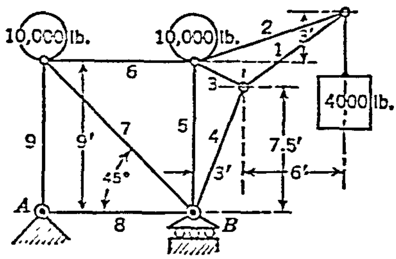


Ans. $R_a = 5600$ lbs. (to left); $R_b = 8200$ lbs. ($\phi = 47°$).
The forces in the members are:

Member No.	1	2	3	4
Force, lbs.	+10,400	−14,900	−14,200	−2500

Member No.	5	6
Force, lbs.	+7100	+5600

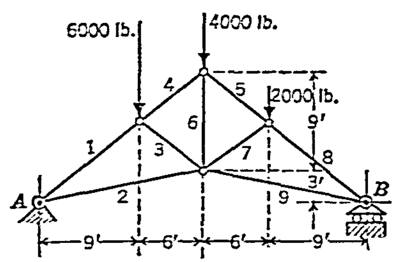
175. The crane structure shown in the sketch carries several loads as indicated. Find the reactions of the supports and the forces acting in all the members.



Ans. R_A is vertical. $R_A = 6000$ lbs.; $R_B = 18,000$ lbs.
The forces in the members are:

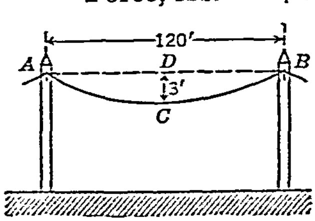
Member No.	1	2	3	4	5
Force, lbs.	−12,000	+10,100	−6250	−10,800	−4000
Member No.	6	7	8	9	
Force, lbs.	+4000	−5600	0	−6000	

176. The roof truss shown in the sketch carries three loads as indicated. Find the reactions of the supports and the forces acting in the members.



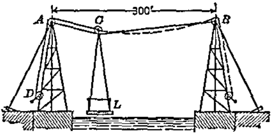
Ans. Reaction R_A is vertical. $R_A = 6800$ lbs.; $R_B = 5200$ lbs.

Member No.	1	2	3	4	5
Force, lbs.	−14,600	+11,600	−4900	−9500	−9500
Member No.	6	7	8	9	
Force, lbs.	+7800	−1600	−11,200	+8900	



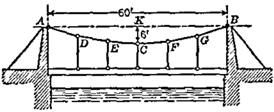
177. An electric cable ACB weighing 80 lbs. and stretched between two poles forms a flat curve with a sag at the middle $CD = 3$ ft. The distance $AB = 120$ ft. Find the tension T_c at the middle of the cable and the tensions

T_a and T_b at its two ends, assuming that the weight of each half of the cable is concentrated at a point 30 ft. from the end of the cable. *Ans.* $T_c = 400$ lbs.; $T_b = T_a = 402$ lbs.



178. A cable car L weighing 10,000 lbs. and used as a ferry is suspended from a pulley which runs on a 306 ft. steel cable AB. The cable is fixed to the tops of the towers A and B, both of the same height and 300 ft. apart. A cable CAD passing over the pulley at A and wound on a drum D is used to move the car to the left; a similar cable is used to pull the car to the right. Neglecting the cable weight, find the tensions in the cables ACB and DAC when $AC = 60$ ft.

NOTE: The point C moves along an ellipse whose foci are at A and B. A normal to the ellipse at C bisects the angle ACB.



179. A suspension bridge 60 ft. long hangs on two chains. The sag of each chain is 6 ft. The load on the bridge is 800 lbs. per running foot. Find the tension in the chains at the middle point C if the curve $ADECFGB$ is a parabola. *Ans.* 30,000 lbs. (approx.).

180. A rope is supported at two points on the same level 30 ft. apart, and its lowest point is 3 ft. below the level of the supports. If the load carried is 15 lbs. per foot (measured horizontally), what are the tensions in the rope at the supports and at its lowest point? What is the slope of the rope at the support?

Ans. $T = 606$ lbs.; $H = 562.5$ lbs.; $\theta_H = 21° 50'$.

181. A cable 150 ft. in length is suspended between two points in a horizontal plane which are 148 ft. apart. If the cable carries a load that is uniformly distributed (measured horizontally), what is the sag of the cable?

186. A cable weighing 12 lb. per lineal foot is suspended between two points at the same elevation 1000 ft. apart. The sag is 120 ft. Calculate the tensions at the lowest point of the cable and at the supports. *Ans.* $T = 12{,}600$ lbs.; $T' = 14{,}000$ lbs.

187. Calculate the length of the cable described in the problem above, using the data obtained in that problem.

Ans. $L = 1032$ ft.

188. A wire rope, 500 ft. long, weighing 1.25 lbs. per lineal foot, is suspended between two supports at the same elevation, 400 ft. apart. Calculate the sag and the maximum tension in the rope.

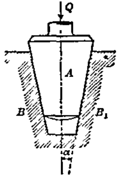
Ans. $f = 134$ ft.; $T = 378$ lbs.

5. Friction.

189. Find the angle of repose for a certain kind of earth whose coefficient of friction is $f = 0.8$.

NOTE: The angle of repose is the steepest slope on which a particle of the earth can rest without slipping downward.

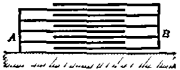
Ans. $\alpha = 38° 40'$.



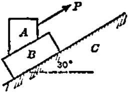
190. A wedge A with a taper of 0.1 in. per inch length is driven into a slot BB_1 by a force $Q = 12{,}000$ lbs. Find the normal force on the faces of the wedge and the force P necessary to pull out the wedge. The coefficient of friction $f = 0.1$.

Ans. $N = 40{,}000$ lbs.; $P = 4000$ lbs.

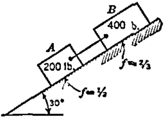
191. Two hundred sheets of paper each weighing 6 grams, filed as shown in the sketch, are glued together alternately so as to form bundles A and B. The coefficient of friction between the sheets and between paper and table is 0.2. If one bundle is held immovable, what is the smallest horizontal force P necessary to pull out the other bundle?



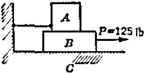
Ans. (1) A immovable, B pulled, $P = 24.12$ Kg.
(2) B immovable, A pulled, $P = 23.88$ Kg.



195 Body A weighs 100 lbs., B weighs 200 lbs., and the coefficients of friction are for A and B, $f = 0.6$, and for B and C, $f = 0.2$ What force P will prevent either block from slipping downward? What force P will give impending motion up the plane? Can body B have impending motion up the plane? (*a*) Solve algebraically (*b*) Solve graphically

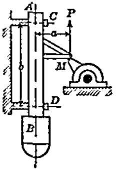


196 The two bodies, having the weights indicated, are connected by a cord and have coefficients of friction on the inclined plane as shown What is the tension in the cord? What is the magnitude and sense of the friction acting on each body? Does the system move or remain at rest? *Ans* $T = 13.4$ lbs

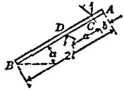


197. Body A weighs 100 lbs and B weighs 200 lbs The coefficient of friction f for A and B is $\frac{1}{4}$ If motion of B is impending when $P = 125$ lbs., what is the coefficient of friction for B and C? What is the tension in the cord?

Ans $f = \frac{1}{3}$, $T = 25$ lbs

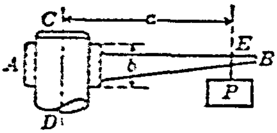


198 The ram AB weighing 300 lbs is operated by a cam mounted on a rotating shaft The distance between the guides C and D is $b = 4.5$ ft The distance between the center line of the ram and the contact point of the cam on shoulder M is $a = 0.45$ ft at the moment of dropping The coefficient of friction in the guides is 0.15 Find the force P necessary to lift the hammer *Ans* $P = 309$ lbs

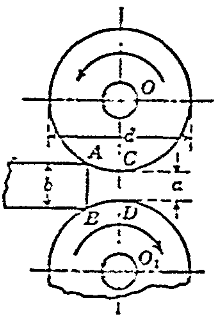


199 A bar AB is held by two supports C and D $CD = a$, $AC = b$ The coefficient of friction of the bar on the supports is f The bar is inclined at an angle α to the horizontal Neglecting the thickness of the bar, how long must it be to be in equilibrium?

Ans $2l \geqq 2b + a + (a/f) \tan \alpha$.

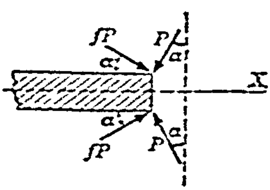


200. A horizontal arm AB has a bushing of length $b = 1$ inch at A. It is slipped over a vertical rod CD. The coefficient of friction between the bushing and the rod is $f = 0.1$. A weight P is suspended at E, a distance a from the center line of the rod. Neglecting the weight of the arm, find the value of a at which the action of the weight P will keep the arm from sliding down the rod. *Ans.* $a \geqq 5$ in.



201. A rolling mill consists of two 20 in. diameter rolls rotating in opposite directions. The distance between the rolls is $a = 0.2$ inches. The coefficient of friction between the rolls and hot iron is $f = 0.1$. What is the maximum thickness b of a bar which can be rolled in this mill?

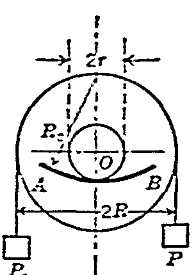
NOTE: In order to pull the bar through the roll, the resultant of all forces on the bar should be directed toward the right.



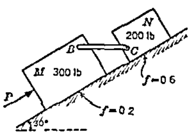
Solution:

The bar is acted upon by the normal forces P between the rolls and the bar, and by the friction forces of P in the direction of motion. Taking the horizontal components of the forces, $\Sigma F_x = 2fP \cos \alpha - 2P \sin \alpha \geqq 0$, or $\tan \alpha \leqq f$. From geometric considerations (with $\tan \alpha = \alpha = \sin \alpha$, since α is small) $(b-a)/2 = (d\alpha^2)/4$. But $\alpha \leqq f$; $b - a \leqq (f^2d)/2$; $b \leqq a + (f^2d)/2$; $b \leqq 0.3$ in.

202. A pulley of radius R has a shaft of radius r through its center. The two ends of the shaft roll on two cylindrical surfaces AB, the radii of which are R_0. The weight of the pulley and shaft is W and the coefficient of friction between the shaft and AB is f. A string thrown over the pulley has weights P_1 and P attached to the ends. $P > P_1$. Find the minimum value of P_1 at which equilibrium will be possible.

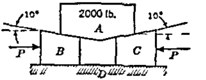


Ans. $P_1 = \dfrac{P(R\sqrt{1+f^2} - fr) - frW}{R\sqrt{1+f^2} + fr}$.



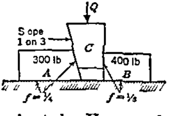
203 Two boxes M and N are connected by the horizontal bar BC The coefficients of friction are shown Neglect the friction in the pins at B and C. (a) What force P will just *prevent the boxes slipping down the chute?* (b) What force P will cause impending motion up the chute?

Ans (a) $P = 95$ lbs , (b) $P = 478$ lbs



204. The stone block A, weighing 2000 lbs , is raised or lowered by means of two wedges B and C The coefficient of friction for the surfaces A–B and A–C is $f = \frac{1}{5}$, f for the surfaces B–D and C–D is $\frac{1}{4}$ What forces P are required to raise the block A? What forces (P reversed) are required to lower the load? (a) Solve algebraically (b) Solve graphically

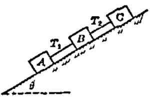
Ans To raise, $P = 641$ lbs To lower, $P = 273$ lbs



205 The wedge C is inserted between two blocks A and B, which rest upon *a rough horizontal plane* One face of the wedge is vertical, the other has a slope of 1 to 3 The coefficients of friction at the various surfaces are indicated How much force Q must be applied to the wedge to start one of the blocks? Which block will move? What is the friction force under the other block when the first one is about to start?

Ans $Q = 69.7$ lbs

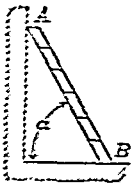
206. Three bodies A, B, and C, weighing 10, 30, and 60 lbs respectively, rest upon a plane inclined at an angle θ with the horizontal The bodies are connected by cords as shown If the coefficients of friction are



for A	$f = 0.1$,
for B	$f = 0.25$,
for C	$f = 0.50$,

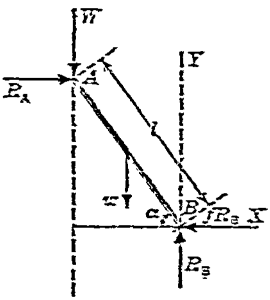
determine the angle θ for impending motion down the plane. Also find the tensions T_1 and T_2 in the cords for this condition.

Ans. $\theta = 21° 5'$; $T_1 = 2.66$ lbs.; $T_2 = 6.45$ lbs.



207. A ladder AB of weight w stands on a rough floor and leans against a smooth wall at A. The coefficient of friction at B is f. What is the minimum value of the angle α at which a man of weight W can climb safely to the top of the ladder?

Solution:



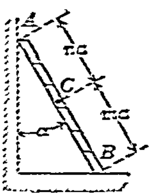
The ladder is safe when the reaction R_A is not larger than the friction force fR_B; i.e., $\Sigma F_x \leqq 0$. From the equilibrium of AB (§ 21), with axes chosen as shown,

$$\Sigma F_y = R_B - W - w = 0, \quad R_B = W + w,$$

$$\Sigma M_B = Wl \cos \alpha + w\frac{l \cos \alpha}{2} - R_A l \sin \alpha = 0.$$

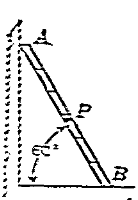
These equations, with $fR_B \leqq R_A$, give

$$f(W + w) \geqq \frac{2W + w}{2 \tan \alpha} \quad \text{or} \quad \tan \alpha \geqq \frac{2W + w}{2f(W + w)}.$$



208. A ladder AB stands on a floor and leans against a wall. The coefficient of friction between the ladder and wall is f_1; between the ladder and floor it is f_2. The total weight of the ladder and of a man standing on it is W. It can be considered concentrated at C, a point dividing the ladder into a ratio of m to n. Find the maximum angle α between the ladder and the wall at which equilibrium is possible. Find the reactions of the wall and of the floor at this angle.

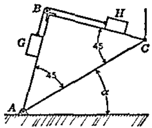
$$\textit{Ans.} \quad \tan \alpha = \frac{f_2(m + n)}{m - nf_1f_2}; \; R_A = \frac{Wf_2}{1 + f_1f_2}; \; R_B = \frac{W}{1 + f_1f_2}.$$



209. A ladder AB stands on a floor and leans against a wall, frictional forces acting at both ends. The ladder carries a load P; the weight of the ladder may be neglected. The angle of friction of the ladder against the wall and on the floor is 15°. Find the maximum distance BP at which equilibrium exists.

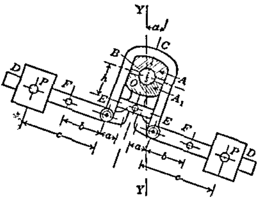
NOTE: The coefficient of friction equals the tangent of the angle of friction.

Ans. $BP = \frac{1}{2}AB$.

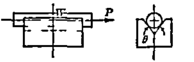


210 A prism ABC is hinged at A Two bodies H and G of equal weight W rest on its sides AB and CB H and G are connected by a string which passes over a pulley B The coefficient of friction between the bodies and the prism is f Angles $BAC = BCA = 45°$ Find the angle α between side AC and the horizontal at which G will start to slide down *Ans* $\alpha = \tan^{-1} f$

211 An apparatus for obtaining coefficients of friction consists of a split bearing AA_1 mounted on a horizontal rotating journal B, the diameter of which is $d = 5$ in The two halves of the bearing are pressed against the journal by means of a yoke C and two levers D and D_1 The short ends of the levers are pressed against the bottom half of the bearing by means of weights P $a = 1\ 5$ in The whole mechanism —bearings yoke, levers, and weights—weighs $W = 80$ lbs Its center of gravity is $h = 6$ in below the center line of the journal Each lever weighs $w_p = 14$ lbs , their centers of gravity at F and F_1 are $b = 25\ 5$ in from the fulcrums E The weights P, each 16 lbs act at distances $c = 45$ in from E The bottom half of the bearings weighs $w_q = 12$ lbs When the journal rotates the axis YY turns through an angle $\alpha = 5°$ Find the coefficient of friction between the journal and the bearing *Ans* $f = 0\ 0057$

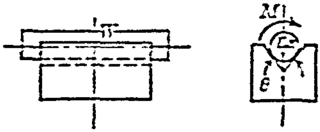


212 The coefficient of friction between the cylinder and the guide is f The cylinder weighs W lbs (a) What is the magnitude of the force P which will just start the cylinder moving horizontally? (b) If P is $2W$, what is the angle θ?



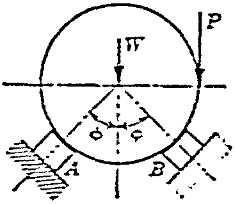
$$Ans \quad (a)\ P = \frac{Wf}{\sin \theta/2},\ (b)\ \sin \frac{\theta}{2} = \frac{f}{2}$$

213. The coefficient of friction between the cylinder and the guide is f. What is the magnitude of the moment M which will just start the cylinder rotating on the guide?

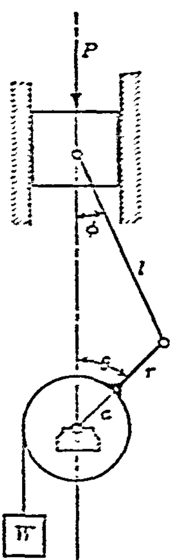


Ans. $$M = \frac{fWr}{(f^2 + 1) \sin \theta/2}.$$

214. A cylinder weighing W lbs. rests on two supports A and B symmetrically spaced about the vertical center line. The coefficient of friction between the cylinder and the supports is f. (*a*) What is the magnitude of the tangential force P necessary to start the cylinder rotating? (*b*) For a certain value of the angles ϕ, this device is self-locking. What is the angle ϕ for which P would have to be infinite in magnitude to start rotation?



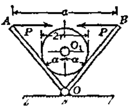
Ans. $$P = \frac{fW}{(1 + f^2) \cos \phi - f}; \quad \phi = \cos^{-1} \frac{f}{1 + f^2}.$$



215. Neglecting friction between the slide block and the groove and in all the pins and bearings of this crank mechanism in the position when the angles are θ and ϕ as shown, what is the magnitude of P necessary to support W? If the coefficient of friction between the slide block and the groove is f, what are the minimum and maximum values of P which will hold W immovable?

Ans. $$P = \frac{Wa(\cos \phi \pm f \sin \phi)}{r \sin (\phi + \theta)}.$$

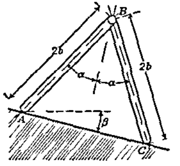
216. A cylinder O_1 rests between two plates hinged together at O. The axis of the cylinder is parallel to the axis of the hinge.



The plates compress the cylinder under the action of two equal and opposite horizontal forces P applied at A and B The weight of the cylinder is W, its radius r The coefficient of friction between plates and cylinder is f, the angle $AOB = 2\alpha$, $AB = a$ Under what conditions will the cylinder be in equilibrium?

Ans $$\frac{W}{\sin\alpha + f\cos\alpha}\;\frac{r}{a} \leqq P \leqq \frac{W}{\sin\alpha - f\cos\alpha}\;\frac{r}{a}$$

An upper limit does not exist for $\tan\alpha < f$



217 A step ladder consists of two identical parts hinged together at B It stands on a plane inclined at an angle β to the horizontal The coefficient of friction between the foot of the ladder and the inclined plane is f, $f > \tan\beta$ Neglecting friction in the hinge B find

1 The range of values of the angle 2α within which the ladder remains in stable equilibrium

2 The conditions under which the frictional force at A will equal zero while the ladder is in equilibrium

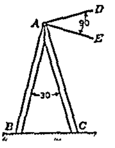
NOTE For small values of 2α the ladder may tip over the point C, for large values of 2α the two parts of the ladder may slide apart

Ans 1 $\tan\alpha \leqq f - \tan\beta + \sqrt{(f - \tan\beta)^2 + f\tan\beta}$

2 $\tan\alpha = 2\tan\beta$

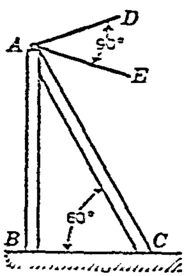
STATICS IN SPACE

6 Concurrent Forces



218 A corner post consisting of two equally inclined timbers AB and AC, joined together at the top, supports two horizontal cables AD and AE Angle $BAC = 30°$ and angle $DAE = 90°$ The tension in each cable is 200 lbs The plane BAC bisects angle DAE Find the forces in the timbers neglecting the effects of their own weight

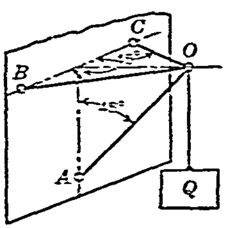
Ans $S_B = -S_C = 200(1 + \sqrt{3}) = 546$ lbs



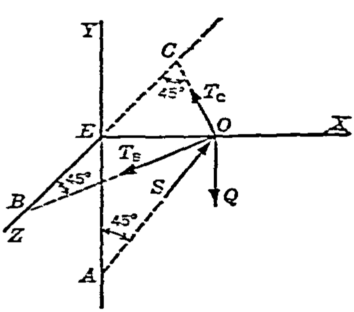
219. Two horizontal telegraph wires AD and AE are attached to the top of a vertical pole AB which is braced by the support AC. Angle $DAE = 90°$. The tension in AD is 24 lbs. and the tension in AE is 32 lbs. Find the angle α between the planes BAC and BAE at which the pole and its support will not be bent out of their plane. Find the force S in the brace if it makes an angle of 60° to the horizontal.

Ans. $\alpha = \tan^{-1}\frac{3}{4}$; $R = 40$ lbs.; $S = 80$ lbs.

220. A captive balloon is held by two ropes which include an angle of 90°. A horizontal wind blowing against the balloon causes the plane of the ropes to make an angle of 60° with the horizontal plane. The direction of the wind is perpendicular to the intersection of the plane of the ropes and the horizontal plane. The inflated balloon weighs 500 lbs.; its volume is 7500 cu. ft. Assume the surrounding air to weigh 0.081 lb. per cu. ft. Find the tensions in the ropes and the force of the wind against the balloon. *Ans.* $R = 124.1$ lbs.; $T = 87.6$ lbs.



221. A weight $Q = 100$ lbs. is suspended from the upper end of a rod AO which is hinged to a wall and supported by two horizontal chains of equal length BO and CO. The rod is inclined 45° to the horizontal. $\angle CBO = \angle OCB = 45°$. Find the force S in the rod and tension T in the chains.



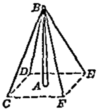
Solution:

The four forces acting on the point O are in equilibrium (§ 23a). From symmetry consideration, the tensions in the chains are equal in magnitude (also $\Sigma F_z = T_C \sin 45° - T_B \sin 45° = 0$; or $T_C = T_B$).

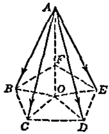
$$\Sigma F_x = S \cos 45° - 2T \cos 45° = 0,$$
$$\Sigma F_y = S \cos 45° - Q = 0.$$

Solving these equations, we find

$$S = \frac{Q}{\cos 45°} = 141 \text{ lbs.}; \; T = \frac{S}{2} = 71 \text{ lbs.}$$

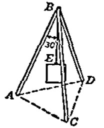


222. A mast is held in a vertical position by four symmetrically located guy ropes. The angle between each pair of adjacent ropes is 60°. Find the force of the mast against the earth, if the tension in each cable is 200 lbs. and the weight of the mast is 400 lbs. *Ans.* $F = 966$ lbs.



223. The four edges AB, AC, AD, AE of a regular pentagonal pyramid represent in value and in direction four forces to the scale 1 ft. = 1 lb. The altitude of the pyramid is 10 ft. and the radius of a circle circumscribing the base is 4.5 ft. Find the resultant force R and the distance x from O to the point at which it intersects the base.

Ans. $R = 40.25$ lbs.; $x = 1.125$ ft.



224. A weight $E = 10$ lbs. is suspended on a rope from the top of a tripod $ABCD$. The legs are of equal length, stand on a horizontal floor and form equal angles with each other. The angle between each leg and the rope is 30°. Find the force S in each leg. *Ans.* $S = 3.85$ lbs.

225. A tripod stands on a smooth floor. The lower ends of its legs are tied together by means of a string so that the legs and strings form a regular tetrahedron. A load P is suspended from the top of the tripod. Find the reactions R of the floor on the feet of the tripod and the tension T in the string.

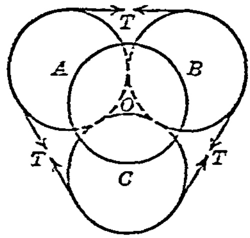
Ans. $R = P/3$; $T = \dfrac{P}{3\sqrt{6}}$.

226. Solve the previous problem assuming the legs to be tied together at their midpoints rather than at their ends. Take into consideration the weight p of each leg, assuming it to be concentrated at the midpoint.

Ans. $R = \dfrac{P + 3p}{3} = p + \dfrac{P}{3}$; $T = \dfrac{2P + 3p\sqrt{6}}{18}$.

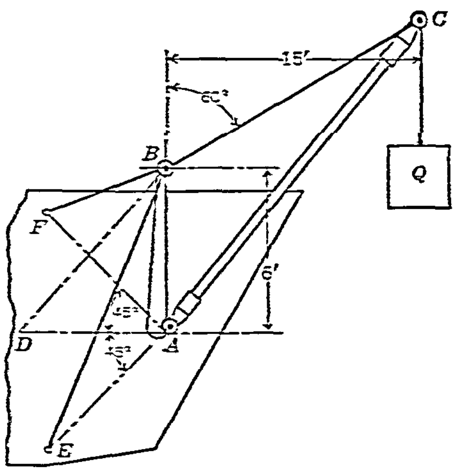
227. Four balls O, A, B, and C of equal radius 2 inches and each weighing 20 lbs. form a pyramid. A, B, and C lie on a

smooth floor touching each other, tied together by means of a string around their equatorial plane. O rests on top of the



others. Find the tension in the string caused by the weight of O.

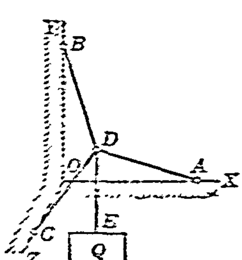
Ans. $T = 2.72$ lbs.



228. A portable crane shown in the sketch carries a weight $Q = 4000$ lbs. $AB = AE = AF = 6$ ft.; angle $EAF = 90°$. The plane of the boom and the vertical column bisects the dihedral angle $EABF$. Neglecting the weights of the crane parts, find the compressive force P_1 in the vertical column AB and find the tensions P_2, P_3, and P_4 in the chain BC and in the cables BE and BF.

Ans. $P_1 = 8380$ lbs.; $P_2 = 11{,}520$ lbs.; $P_3 = P_4 = 10{,}000$ lbs.

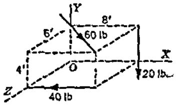
229. Strings AD, BD, and CD are fixed at points A, B, and C, each at a distance l from the origin O. They are tied together at D. $BD = AD = CD = L$. The coordinates of the point D are $x = y = z = \frac{1}{3}(l - \sqrt{3L^2 - 2l^2})$. A load Q is suspended at D. Find the tensions T_1, T_2, and T_3 of the strings, when $L < l$.



$$Ans.\quad T_1 = T_2 = Q\frac{L(l - \sqrt{3L^2 - 2l^2})}{3l\sqrt{3L^2 - 2l^2}};$$

$$T_3 = Q\frac{L(l + 2\sqrt{3L^2 - 2l^2})}{3l\sqrt{3L^2 - 2l^2}}.$$

7. Reduction of a System of Forces to a Simpler Form.



230. The box in the sketch is acted upon by three forces, two along the edges and the other along a diagonal, as shown. Reduce these forces to a system consisting of a single force, acting through point O, and a couple.

Solution.

Each of the given forces is resolved into a single force acting through point O and a couple (§ 27) These are then combined into a single force through O and a couple The calculations are shown in tabular form·

F	F_x	F_y	F_z	M_x	M_y	M_z
20	—	−20	—	—	—	−160
40	−40	—	—	—	−240	—
60	+48	—	+36	+144	—	−192
	+ 8	*−20*	*+36*	*+144*	*−240*	*−352*

$$R = \sqrt{8^2 + 20^2 + 36^2} = \mathit{42\ lbs},$$

$$\alpha = \cos^{-1}\frac{8}{42} = 79°\,0',$$

$$\beta = \cos^{-1}\frac{-20}{42} = 118°\,25',$$

$$\gamma = \cos^{-1}\frac{36}{42} = 31°\,0',$$

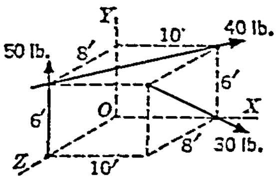
$$C = \sqrt{144^2 + 240^2 + 352^2} = \mathit{450\ lbs\text{-}ft},$$

$$\alpha' = \cos^{-1}\frac{144}{450} = 71°\,20',$$

$$\beta' = \cos^{-1}\frac{-240}{450} = 122°\,15',$$

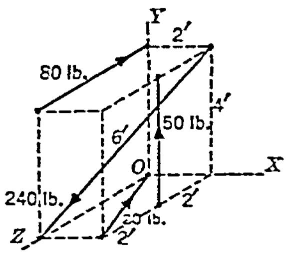
$$\gamma' = \cos^{-1}\frac{-352}{450} = 141°\,30'$$

Ans The equivalent system consists of a single force of 42 lbs passing through the origin O, making angles of 79° 0′, 118° 25′, and 31° 0′ with the OX, OY, and OZ axes, respectively, and a couple of 450 lbs -ft whose vector makes the angles 71° 20′, 122° 15′, and 141° 30′ with the OX, OY, and OZ axes, respectively The couple lies in a plane normal to its vector and acts clockwise when viewed in the direction in which the vector points



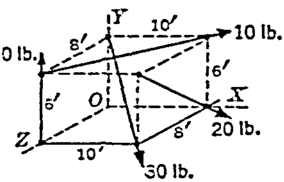
231. A box 10 ft. by 6 ft. by 8 ft. is acted upon by three forces, as shown in the diagram. What single force, acting through the origin O, and couple, are equivalent to the force system shown.

Ans. R = 66 lbs.; C = 826 lbs.-ft.



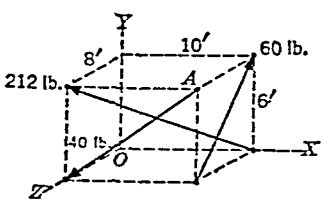
232. Replace the system of four forces shown by a single force acting through O, and a couple.

Ans. R = 153 lbs.; C = 401 lbs.-ft.



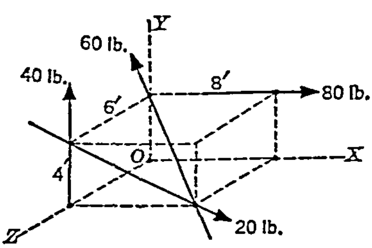
233. The rectangular parallelopiped shown is acted upon by four forces whose magnitudes and directions are indicated. What single force, acting through the origin O, and couple, are equivalent to the force system shown?

Ans. R = 39 lbs.; C = 498 lbs.-ft.



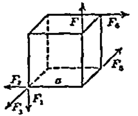
234. (a) Replace the three forces shown in the diagram by an equivalent system consisting of a single force, acting through point O, and a couple. (b) Replace the force system shown in the diagram by a single force, acting through the point A, and a couple.

Ans. (a) R = 225 lbs.; C = 1530 lbs.-ft.



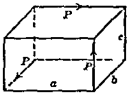
235. Replace the force system shown in the diagram by a single force, acting through the origin, and a couple.

Ans. R = 83 lbs.; C = 401 lbs.-ft.

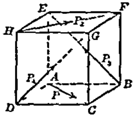


236 Forces are applied to the vertices of a cube as shown in the sketch Find the conditions under which the cube will be in equilibrium

Ans $F_1 = F_2 = F_3 = F_4 = F_5 = F_6$



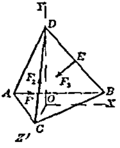
237 Three equal forces P act along three edges of a parallelopiped which do not intersect What relation must exist between the lengths of the edges a, b, and c in order that the system of forces may be reduced to one resultant force *Ans* $b = a - c$.



238 Four forces $P_1 = P_2 = P_3 = P_4 = P$ are applied to the four vertices A, B, D, and H of a cube The force P_1 acts along the diagonal AC, P_2 acts along HF, P_3 acts along BE, and P_4 acts along DG Reduce the system of forces to its simplest form

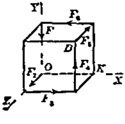
Ans Single force $2P$ acting along line DG

239 A regular tetrahedron is loaded with several forces F_1 along the edge AB, F_2 along CD, and F_3 in the point E, the middle of the edge BD The values of F_1 and F_2 are arbitrary, the projections of F_3 on the axes OX, OY, OZ are $-(F_2/2)$, $+F_2 5/6\sqrt{3}$, $-F_2\sqrt{2/3}$ Can this system of forces be reduced to one resultant and if so, find the intersection of this resultant with the plane YOZ



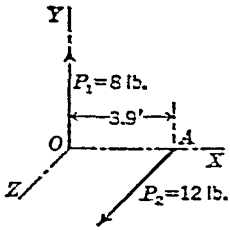
Ans Yes The coordinates are

$$z = -\frac{M_y}{R_x} = -\frac{a\sqrt{3}(F_1 + F_2)}{6F_1 - 3F_2}, \; y = 0$$

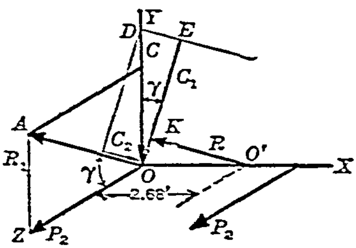


240 Six equal forces of 4 lbs each are applied to the vertices of a cube, the sides of which are $a = 2$ inches long Reduce this system of forces to its simplest form

Ans A couple of $16\sqrt{3}$ lb in, vector directed along the diagonal to K

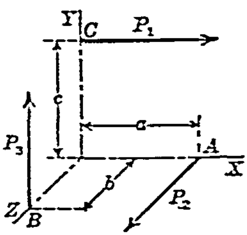


241. A system of two forces consists of $P_1 = 8$ lbs. acting along OY and $P_2 = 12$ lbs. acting parallel to OZ. $OA = 3.9$ ft. Reduce the system to a "canonical form." Find the angles α, β, and γ which the central axis makes with the coordinate axes.



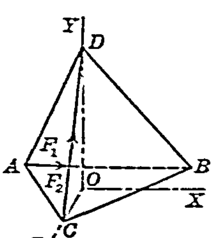
Solution:

Following the procedure recommended in § 31*a*, the resultant force R is equal to $\overline{OA}$ = *14.4 lbs.*; its direction is determined by $\alpha = 90°; \beta = \tan^{-1}\frac{3}{2}; \gamma = \tan^{-1}\frac{2}{3}$. The resultant moment components are $C_z = 0$; $C_x = 0$; $C_y = -3.9P_2$; $C = 3.9P_2$, is represented by vector DO. Component couple $EO = C_1 = C\cos\gamma$ and the force R (vector OA) can be replaced (§ 14) by a single force $O'K = R$, parallel to OA, and whose line of action (central axis) is at a distance $OO' = C/R$ = *2.68 ft.* from O. The system is now reduced to force $O'K$ equal to 14.4 lbs. and the remaining couple $C_2 = C\sin\gamma$ = *25.9 lbs.-ft.*, acting in a plane perpendicular to $O'K$.



242. Three forces P_1, P_2, and P_3 are parallel to the coordinate axes, as shown in the sketch. The points of application are A, B, and C at distances a, b, and c from the origin O. (*a*) What is the condition under which the forces may be reduced to one resultant? (*b*) What is the condition under which there exists a central axis passing through the origin?

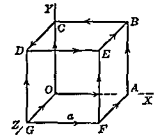
Ans. (*a*) When $\dfrac{a}{P_1} + \dfrac{b}{P_2} + \dfrac{c}{P_3} = 0$; (*b*) when $\dfrac{P_1}{bP_3} = \dfrac{P_2}{cP_1} = \dfrac{P_3}{aP_2}$.



243. A regular tetrahedron $ABCD$ has edges of length a. F_1 is applied to A along AB and F_2 to C along CD. Find the coordinates x and y of the intersection of the central axis with the plane XOZ.

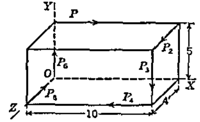
Ans. $x = -\dfrac{4F_1F_2}{3\sqrt{6}(F_1^2 + F_2^2)}a;$

$$z = \frac{\sqrt{3}}{6} \times \frac{2F_2^2 - F_1^2}{(F_1^2 + F_2^2)}a.$$



244 Twelve equal forces P are applied to the corners of a cube whose edges are a in length, as shown in the sketch Reduce this system of forces to a canonical form

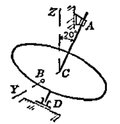
Ans $R = 2P\sqrt{6}$, $\cos\alpha = \frac{1}{6}\sqrt{6}$, $\cos\beta = \frac{1}{3}\sqrt{6}$, $\cos\gamma = -\frac{1}{6}\sqrt{6}$, $M = \frac{1}{3}aP\sqrt{6}$



245 Six forces act along the edges of a parallelopiped with sides 10, 4, and 5 ft long, as shown in the sketch $P_1 = 4$ lbs , $P_2 = 6$ lbs , $P_3 = 3$ lbs , $P_4 = 2$ lbs , $P_5 = 6$ lbs , $P_6 = 8$ lbs Reduce this system of forces to the canonical form and find the coordinates x and z of the intersection of the central axis with the plane XOZ

Ans $R = 5\ 38$ lbs , $\cos\alpha = 0\ 37$, $\cos\beta = 0\ 93$, $\cos\gamma = 0$
$M = 47\ 6$ lbs ft , $x = -10$ ft , $z = -11\ 9$ ft

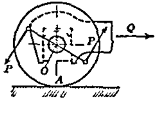
8 Equilibrium of a Rigid Body



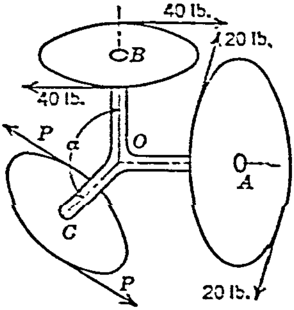
246 A treadmill consisting of a round turntable mounted rigidly on a shaft AD inclined at 20° to the vertical is turned by a horse weighing 800 lbs , who always remains at B on the horizontal radius CB $CB = 9$ ft Find the turning moment around the axis of rotation

Ans $M = 2460$ lbs -ft

247 A windmill has four blades inclined at an angle $\alpha = 15°$ to the plane normal to the axis of rotation The force of the wind on each blade is 200 lbs , it acts normally to the plane of the blade and is applied at a point 9 ft from the axis of rotation Find the turning moment *Ans* $M = 1865$ lbs -ft

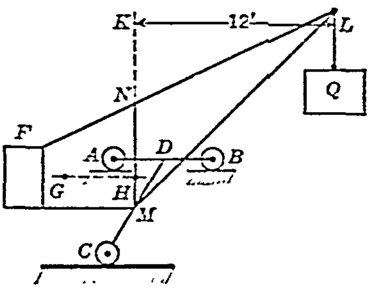


248 An electric motor mounted on the axle of a street car exerts a torque of 3600 lbs ft The radius of the wheels is 1 8 ft Find the tractive effort at the rail, assuming that the car is standing on horizontal track *Ans* $Q = 2000$ lbs



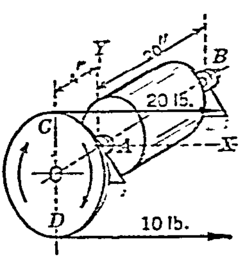
249. Three couples of forces 20 lbs., 40 lbs., and P lbs. are applied to the circumferences of three discs with diameters of 12 in., 8 in., and 4 in. respectively. The axes OA, OB, and OC are all in one plane; $AOB = 90°$. Find the magnitude of P and the angle $BOC = \alpha$ for which the system will be in equilibrium.

Ans. $P = 100$ lbs.; $\alpha = 180° - \tan^{-1} 0.75$.



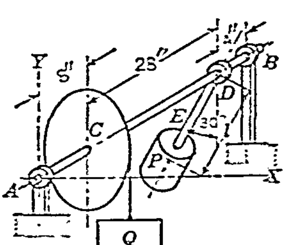
250. A crane is mounted on a three-wheel car ABC. $AD = BD = 3$ ft.; $CD = 4.5$ ft.; $CM = 3$ ft.; $KL = 12$ ft. The crane is counterbalanced by a weight F. The weight of the crane including the counterweight is 20,000 lbs., and acts at G, a point in the plane $LMNF$ at a distance $GH = 1.5$ ft. from the axis MN. The load Q is 6000 lbs. Find the forces on the wheels when the plane LMN is parallel to AB.

Ans. $N_A = 1666$ lbs.; $N_B = 15{,}667$ lbs.; $N_C = 8667$ lbs.



251. The belt pulley of a generator is 8 inches in diameter. The dimensions of the shaft are given in the sketch. The tight side of the belt has a tension of 20 lbs., the slack side—10 lbs. Find the torque M and the reactions of the bearings due to the belt pull.

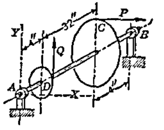
Ans. $M = 40$ lbs.-in.;
$R_A = 36$ lbs. to left; $R_B = 6$ lbs.



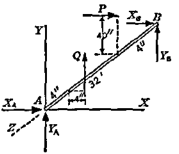
252. A horizontal shaft resting on two bearings A and B carries a pulley C of 16 inches diameter which is loaded by a weight $Q = 50$ lbs. hanging on a rope. A load $P = 200$ lbs. is rigidly attached to the shaft by a rod DE. $AC = 8$ in.; $CD = 28$ in.; $DB = 4$ in. At equilibrium the rod DE makes an angle of 30° with

the vertical. Find the distance l between the center of gravity of P and the axis of the shaft. Find the bearing reactions.

Ans. $l = 4''$; $R_A = 60$ lbs.; $R_B = 190$ lbs.



253. A horizontal shaft AB carries a gear C of 80 inches diameter and a pinion D of 8 inches diameter. The other dimensions are given in the sketch. A horizontal force $P = 20$ lbs. is applied tangentially to the rim of C; D is loaded tangentially by a vertical force Q. Find the value of Q and the bearing reaction when the system is in equilibrium.



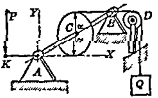
Solution:

The components of the bearing reactions on the journal are assumed to be as shown.

Considering the equilibrium of forces acting on the rotating assembly (§ 28), the following five equations are written:

$$\Sigma F_x = 20 + X_A + X_B = 0,$$
$$\Sigma F_y = Q + Y_A + Y_B = 0,$$
$$C_x = 4Q + 40Y_B = 0,$$
$$C_y = -36P - 40X_B = 0,$$
$$C_z = 4Q - 40P = 0.$$

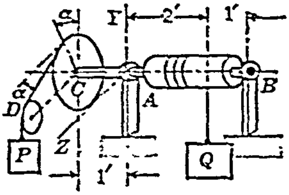
Solving these, we find $Q = 200$ *lbs.*; $X_B = -18$ *lbs.* (acts to left); $Y_B = -20$ *lbs.* (acts down), $X_A = -2$ *lbs.* (acts to left); $Y_B = -180$ *lbs.* (acts down).



254. A workman lifts a load $Q = 160$ lbs. by means of the winch shown in the sketch. The diameter of the drum is 4 inches; the length of the crank $AK = 16$ in.; $AC = CB = 20$ in. Find the vertical force P at the end of the crank and the forces on the bearings when the crank is *horizontal* and the force P vertical, as shown.

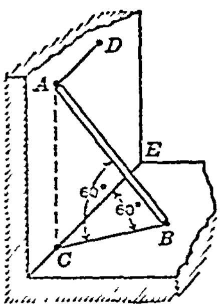
Ans. $P = 20$ lbs.; $R_{XA} = 80$ lbs.; $R_{YA} = 20$ lbs.; $R_{XB} = -80$ lbs.; $R_{YB} = 0$.

255. The drum AB of a winch carries a rope on which a load Q is hanging. The radius of the wheel C rigidly attached to the shaft is 6 times larger than the radius of the drum; the other dimensions are given in the sketch. A rope wound on the rim of the wheel is loaded by a weight $P = 12$ lbs. It leaves the rim at an angle of $\alpha = 30°$ with the horizontal. Find the weight Q for which the winch is in equilibrium; find the reactions at A and B, neglecting the weight of the shaft.



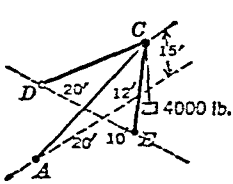
Ans. $Q = 72$ lbs.; $R_{ZA} = -13.9$ lbs.; $R_{YA} = 32$ lbs.; $R_{ZB} = 3.4$ lbs.; $R_{YB} = 46$ lbs.

256. A rod AB is held in position by two horizontal strings AD and BC. At A the rod rests against a vertical wall, to which the rope end D is also attached. At the point B the rod rests on a horizontal floor. A and C are on the same vertical line. The rod weighs 16 lbs. Neglecting friction at A and B, find the tensions T_A and T_B in the strings and the reactions of the wall and the floor.



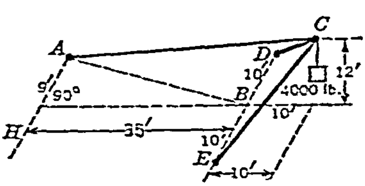
Ans. $T_A = 2.30$ lbs.; $R_A = 4$ lbs.; $T_B = 4.6$ lbs.; $R_B = 16$ lbs.

257. CD and CE are two posts of a crane supporting a load of 4000 lbs. attached at C and held in position by the guy rope AC. Determine all the forces acting at the point C.

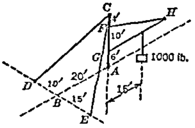


Ans. Force in AC: 5650 lbs., tension; in DC: 3950 lbs.; in CE: 6080 lbs., compression.

258. The sketch represents a shear-legs crane. It consists of two posts CD and CE, hinged together at the top and hinged at their bases in the horizontal plane $DBEHA$. The two posts are held in position by the back-stay AC, which is inclined as shown. The crane

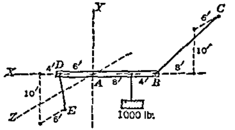


carries a load of 4000 lbs. at C. Determine the force in each of the members AC, CD, and CE.



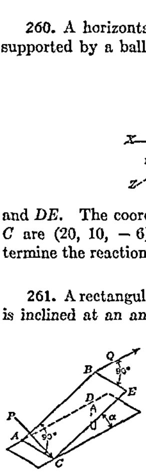
259. The post of this crane rests in a socket at A and is kept from overturning by two unsymmetrical guy ropes CD and CE. The boom GH carries a load of 1000 lbs. and is turned until it is in the plane of ABC. Determine the reaction at A and the tensions in the guy ropes CD and CE.

260. A horizontal bar BD carries a load of 1000 lbs. It is supported by a ball and socket joint at A and by two cables BC

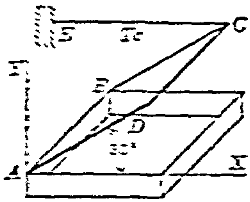


and DE. The coordinates of E are $(-10, -10, -6)$, and of C are $(20, 10, -6)$, the axes being chosen as in sketch. Determine the reaction at A and the tension in each cord.

261. A rectangular plate with sides $AB = 4$ ft. and $AC = 2$ ft. is inclined at an angle $\alpha = 30°$ to the horizontal plane. At D the plate rests on a peg. The corner A is fixed. A horizontal force $Q = 10$ lbs. acts at B in a direction perpendicular to BE. At C a force $P = 8$ lbs. acts downward perpendicular to the surface of the plate. $AD = 3$ ft. Neglecting the weight of the plate, find the distance h between the point D and the edge AC and the reaction at the points A and D.

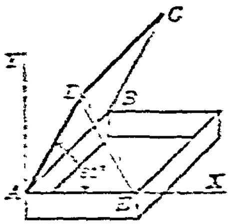


Ans. $h = 2.34$ ft.; $R_A = 9.7$ lbs.; $R_D = 6.5$ lbs.



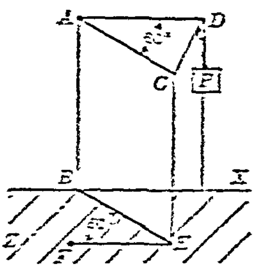
262. A square box lid weighing 4 lbs. can rotate around the axis AB on hinges at A and B. A horizontal rope EC parallel to AX holds the lid at an angle $DAX = 30°$. Find the reactions of the hinges.

Ans. $R_{XA} = 0$; $R_{YA} = 2$ lbs.; $R_{XB} = 3.5$ lbs.; $R_{YB} = 2$ lbs.



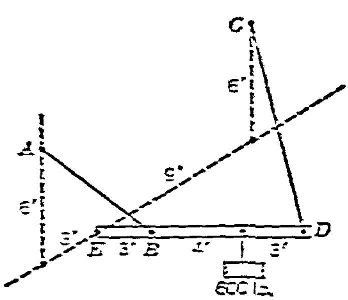
263. The lid $ABCD$ of a rectangular box is supported by a rod DE as shown. The weight of the lid is 24 lbs. $AB = 3$ ft. $AE = AD = 2$ ft. The angle $DAE = 60°$. Find the reactions of the hinges A and B and the force S in the rod. Neglect the weight of the rod.

Ans. $S = 6.92$ lbs.; $R_{XA} = 3.46$ lbs.; $R_{YA} = 6$ lbs.; $R_{XB} = 0$; $R_{YB} = 12$ lbs.

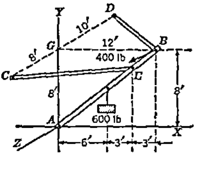


264. A rectangular door rotating around the vertical axis AB is open to an angle 60° and is held in this position by two ropes. One, CD, is passed over a pulley D and carries a weight $P = 20$ lbs. The other, EF, is attached to the floor at F. The door weighs 40 lbs.; $AD = AC = 6$ ft.; $AB = CE = 8$ ft. Find the tension T in the rope FE; also the reactions of the cylindrical hinge at A and of the step bearing at B.

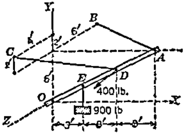
Ans. $T = 20$ lbs.; $R_{XA} = -17.5$ lbs.; $R_{ZA} = 4.3$ lbs.; $R_{XB} = 27.5$ lbs.; $R_{YB} = 40$ lbs.; $R_{ZB} = 13.0$ lbs.



265. The boom DE bears one 600 lb. load, and is held in a horizontal position by a socket at E and cords AB and CD. ACE is a vertical plane perpendicular to the bar. Find the tensions in the cords and the reaction at E.



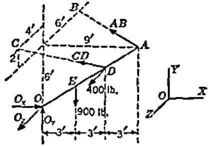
266. The bar AB lies in the XY plane, carries a load of 600 lbs. at its midpoint, and is pulled by a 400 lb. force acting parallel to the Z axis at B. The bar rests in a socket at A and is supported by the two bars BD and CE. The bar BD is horizontal, and the line CD is parallel to the Z axis. Determine the forces in the two bars BD and CE, and determine the axial components of the reaction at A.



267. The bar AO rests in a socket at O and is held in position by bars AB and CD. It is acted upon by a load at E and by a force at D acting parallel to the Z axis. Find the reaction at O and the forces in the two bars AB and CD.

Solution:

The bar AO is in equilibrium under the action of the forces AB, CD, 400 lbs., 900 lbs., and a force at O, which has rectangular components, O_x, O_y, and O_z.



From the geometry of the structure (§ 23):

Length $CD = \sqrt{6^2 + 4^2 + 4^2} = 8.25$ Length $AB = 10.82'$

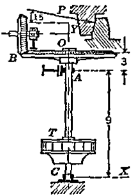
The x-component of CD, $CD_x = \dfrac{6}{8.25} CD$, $AB_x = \dfrac{9}{10.82} AB$,

$$CD_y = \frac{4}{8.25} CD, \qquad AB_y = 0,$$

$$CD_z = \frac{4}{8.25} CD, \qquad AB_z = \frac{6}{10.82} AB.$$

in two cases (1) when the wind acts on all four blades, (2) when the blade D is dismantled and DE is vertical

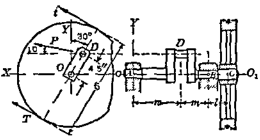
Ans Case 1 $P = 800$ lbs, $R_{XA} = 0$, $R_{YA} = 267$ lbs, $R_{XC} = 0$, $R_{ZC} = 832$ lbs, $R_{YC} = 533$ lbs,
Case 2 $P = 600$ lbs, $R_{XA} = 160$ lbs, $R_{YA} = -78$ lbs, $R_{XC} = -40$ lbs, $R_{ZC} = -624$ lbs, $R_{YC} = 678$ lbs



270 A water turbine T exerts a torque of 720 ft lbs which is balanced by the tooth force of the bevel gear OB and the bearing reactions The tooth force is normal to the radius OB and acts at an angle of 15° to the horizontal The total weight of turbine, shaft, and gear is 2400 lbs The center of gravity of the system lies on the vertical center line OC $OB = 1.8$ ft $AC = 9$ ft and $AO = 3$ ft Find the reactions of the step bearing C and of the sleeve bearing A

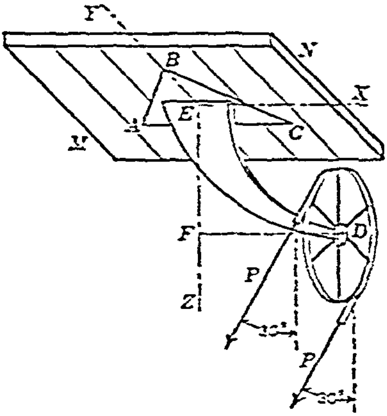
Ans $R_{XA} = 21.4$ lbs, $R_{ZA} = 533$ lbs, $R_{XC} = -21.4$ lbs, $R_{YC} = 2507$ lbs, $R_{ZC} = -133$ lbs

271 The connecting rod of a steam engine exerts a force $P = 4000$ lbs which acts through the center of the crank pin D at an angle of 10° to the horizontal The crank plane ODO_1 is at an angle of 30° to the vertical The flywheel on the crankshaft acts as a pulley and transmits the power to the main line shaft by means of a cable, both sides of which are parallel and extend in a direction 30° to the horizontal The force P is balanced by the tensions T and t in the cable and the bearing reactions at A and B The flywheel weighs 3000 lbs, its diameter is $d = 6$ ft The sum of the tensions $T + t = 1500$ lbs The crank radius is $r = 4\frac{1}{2}$ in, $l = 10$ in, $m = 12$ in, $n = 18$ in Find the reactions of the bearings A and B



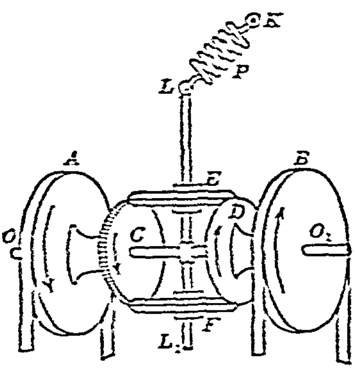
Ans $R_{XA} = -1140$ lbs, $R_{YA} = -1028$ lbs
$R_{XB} = -4096$ lbs, $R_{YB} =$ 2082 lbs

272. A pulley hanger is bolted to the ceiling MN at A and C. Point B rests against the ceiling. ABC forms an equilateral triangle 12 inches on a side. E is the center of ABC. $EF = 16$ in. $FD = 20$ in. EF is perpendicular to the plane of ABC; FD is perpendicular to EF and is parallel to AC; the plane of the pulley is perpendicular to FD. The tension in each side of the belt is 240 lbs. and they leave the pulley at an angle of 30° to the vertical. Find the reactions at A, B, and C. Neglect the weight of the pulley and hanger.



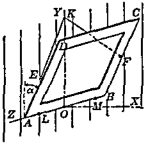
Ans. $R_{YA} = 280$ lbs.; $R_{ZA} = 370$ lbs.; $R_{ZB} = 230$ lbs.; $R_{YC} = -520$ lbs.; $R_{ZC} = -1016$ lbs.

273. A dynamometer built as shown in the sketch measures the torque transmitted from pulley A to pulley B. Both pulleys rotate freely on the fixed axle OO_1. The bevel gears C and D are attached rigidly to A and B, respectively, and they mesh with gears E and F which rotate around the vertical shaft LL_1. Shaft LL_1 can rotate about shaft OO_1 and is held from doing so by the spring balance P fixed at K. The diameters of gears C, D, E, and F are each 8 in.



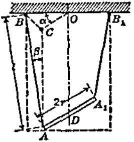
The torque transmitted from A to B is 960 in.-lbs. LK is perpendicular to the plane OLO_1; $LE = 20$ in. Find the forces N exerted by the gears E and F on the shaft LL_1 and the reading on the spring. *Ans.* $N_{FL} = -N_{EL} = 240$ lbs.; $P = 80$ lbs.

274. A rectangular picture hangs on a vertical wall. It is suspended from a hook K by means of a wire FKE. The side AB is 2 ft. long and rests horizontally on two nails at L and M. $AL = MB$. The wire is attached to E and F. $AE = ED = BF$



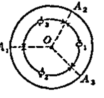
$= FC = 1\frac{1}{4}$ ft. The angle between the wall and the picture is $\tan^{-1} \frac{3}{4}$. The picture weighs 40 lbs. and its center of gravity is in the center of $ABCD$. The wire is 2 ft. 10 in. long. Find the tension T in the wire and the forces on the nails L and M.

Ans. $T = 17$ lbs.; $R_{XL} = R_{XM} = -\ 9$ lbs.; $R_{YL} = R_{YM} = -12$ lbs.

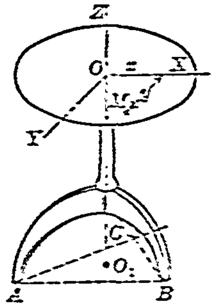


275. A rod AA_1 is suspended on two wires BA and B_1A_1 of equal length, fixed at B and B_1. The length of the rod $AA_1 = BB_1 = 2r$; its weight is P. The rod is turned around a vertical axis through an angle α. Find the moment M of the couple necessary to hold the rod in this position. Find the tension T in the wires.

$$\textit{Ans.}\quad M = \frac{Pr^2 \sin \alpha}{\sqrt{l^2 - 4r^2 \sin^2 \alpha/2}};\quad T = \frac{Pl}{2\sqrt{l^2 - 4r^2 \sin^2 \alpha/2}}.$$



276. A round table $A_1A_2A_3$ stands on three legs A_1, A_2, and A_3. A load is placed in the center O. If the forces in the legs are in the ratio of $1 : 2 : \sqrt{3}$, what are the angles, ϕ_1, ϕ_2, and ϕ_3. *Ans.* $\phi_1 = 150°$; $\phi_2 = 90°$; $\phi_3 = 120°$.



277. A table stands on three legs, the feet of which form an equilateral triangle ABC. Each side of ABC has the length a. The weight of the table is P and its center of gravity is on the vertical OO_1. O_1 is the center of the triangle ABC. A weight p is placed at point M, the coordinates of which are x and y; the axis OX is parallel to AB. Find the force exerted on the floor by each foot.

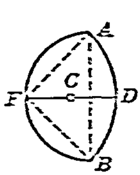
Ans. $N_A = \dfrac{P+p}{3} + p\dfrac{y\sqrt{3} - 3x}{3a}$; $N_B = \dfrac{P+p}{3} + p\dfrac{y\sqrt{3} + 3x}{3a}$;

$$N_C = \frac{P+p}{3} - p\frac{2y\sqrt{3}}{3a}.$$

278. The depth to which the piers of a bridge were sunk below the bottom of a river was calculated on the assumption that the weight of the pier and its load were balanced by the reaction of the ground against the bottom of the pier and the friction of the ground against the sides of the pier. The ground—fine sand, saturated with water—was considered as a liquid. The load on each pier is 323,300 lbs. The pier weighs 5220 lbs. per foot height. It extends 28 ft. above the water level and the water is 21 ft. deep. The area of the bottom of the pier is 38.5 sq. ft.; the side surface is 22 sq. ft. per foot height. The weight of the water-saturated sand is 114.4 lbs. per cu. ft.; water weighs 62.3 lbs. per cu. ft. The coefficient of friction between the iron caisson surrounding the pier and the sand is $f = 0.176$. Find the depth h to which the pier is sunk below the river bottom.

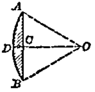
Ans. $h = 40$ ft.

9. Centroid and Center of Gravity.



279. Find the position of the center of gravity C of a wire frame $AFBD$ which consists of the quarter circle ADB of radius $FD = R$ and the semicircle AFB of diameter AB. The wire is uniform in both arcs.

Ans. $CF = R\dfrac{\pi - 2 + 2\sqrt{2}}{\pi(1 + \sqrt{2})}$.

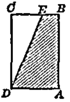


280. Find the position of the center of gravity C of the area of a circular segment ADB. Radius $AO = 30$ in.; the angle AOB is 60°.

Ans. $OC = 27.7$ in.

281. Find the position of the center of gravity of an area bounded by a semicircle AOB of radius R and by two straight lines AD and DB. $OD = 3R$.

Ans. $OC = 1.19R$.



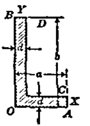
282. Cut a rectangular plate $ABCD$ along the line DE through the corner D in such a way that when the part $ABED$ is suspended at the point E, the side $DA = a$ will be horizontal.

Solution:

The center of gravity of $DEBA$ must be located vertically under E (§ 32). Assuming $CD = h$; $EB = x$, the centroid of $DEBA$ is at the distance x from AB. Considering $ABCD$ as consisting of parts $DEBA$ and CDE (§ 37), we may write

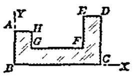
$$(ah)\cdot\frac{a}{2} = \left(\frac{a+x}{2}\;h\right)\;(a-x) + \frac{(a-x)h}{2}\cdot\frac{1}{3}(a-x).$$

Solving, we find $x = 0.366a$.



283. Find the coordinates of the center of gravity of the cross-section of an angle bar as shown in the sketch. $OA = a$; $OB = b$; $AC = BD = d$.

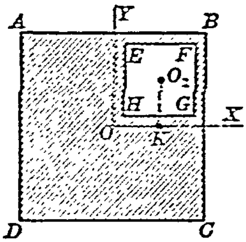
Ans. $\bar{x} = \dfrac{a^2 + bd - d^2}{2(a + b - d)}$; $\bar{y} = \dfrac{b^2 + ad - d^2}{2(b + a - d)}$.



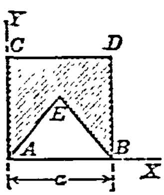
284. Find the center of gravity of a plate having the shape shown in the sketch. $AH = 2$ in.; $HG = 1.5$ in.; $AB = 3$ in.; $BC = 10$ in.; $EF = 4$ in.; $ED = 2$ in.

Ans. $\bar{x} = 5.77$ in.; $\bar{y} = 1.77$ in.

285. A board $ABCD$, 2 ft. square, has a square hole $EFGH$ cut in it as shown. The sides of the hole are parallel to the sides of the board and they are each 0.7 ft. long. O and O_1 are the centers of the two squares. OK and O_1K are parallel to the sides of the squares and $OK = O_1K = 0.5$ ft. Find the coordinates x and y of the center of gravity of the remaining board material.



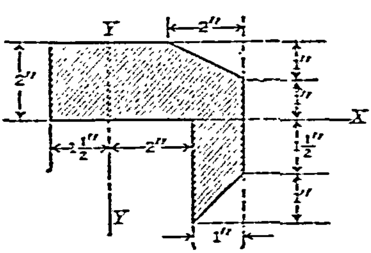
Ans. $\bar{x} = \bar{y} = -0.838$ in.



286. In a square $ABCD$ with sides equal to a in length, find a point E such that it will be the center of gravity of the figure obtained when the isosceles triangle AEB is cut out of the square.

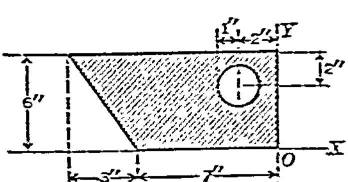
Ans. $\bar{y} = 0.635a$.

287. Four men carry a triangular plate. Two hold vertices of the triangle. The other two hold the two sides forming the third vertex. How far from the third vertex should these men grasp the plate so that each man will carry ¼ of the weight of the plate? *Ans.* At ⅓ of the side length from the vertex.



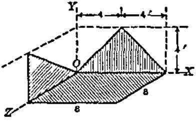
288. Locate the centroid of the shaded area shown.

Ans. $\bar{x} = 0.93$ in.; $\bar{y} = 0.53$ in.

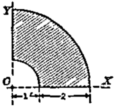


289. Determine the coordinates of the centroid of the shaded area shown.

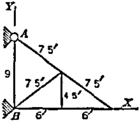
Ans. $\bar{x} = -4.44$ in.; $\bar{y} = 3.12$ in.



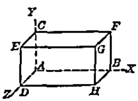
290. A thin plate of tin made up of two triangles and a square has been bent as shown in the figure, the isosceles triangle being in the XY plane, the right triangle in the YZ plane, while the square remains in a horizontal plane. Determine the coordinates of the center of gravity of the plate when bent as specified. *Ans.* $\bar{x} = 3.33$ in., $\bar{y} = 0.444$ in., $\bar{z} = 3.55$ in.



291. Determine the coordinates of the centroid of the quarter ring indicated by the shaded area. *Ans.* $\bar{x} = \bar{y} = 1.38$ in.



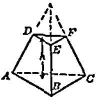
292. Find the coordinates of the center of gravity of a truss made of seven members, as shown in the sketch. The weight per unit length of each member is the same; their lengths are shown in the drawing.
Ans. $\bar{x} = 4.41$ ft., $\bar{y} = 2.82$ ft.



293. Find the center of gravity of a system of weights located at the vertices of the rectangular parallelopiped shown in the sketch. $AB = 20$ in., $AC = 10$ in., $AD = 5$ in. The weights and their positions are

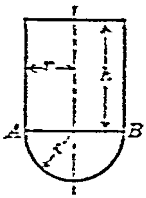
Corner	A	B	C	D	E	F	G	H
Weight	1	2	3	4	5	3	4	3 lbs.

Ans. $\bar{x} = 9.6$ in., $\bar{y} = 6.0$ in., $\bar{z} = 3.2$ in.



294. $ABCDEF$ is the frustrum of a tetrahedron. The area $ABC = a$ and the area $DEF = b$. The altitude of the frustum is h. Find the distance $\bar{y}$ of the center of gravity from the base ABC.

$$Ans.\quad \bar{y} = \frac{h}{4} \times \frac{a + 2\sqrt{ab} + 3b}{a + \sqrt{ab} + b}.$$



295. A body consists of a cylinder of height h mounted on a hemisphere. Both have the radius r. What is the maximum value of h for which the body will remain in stable equilibrium on its hemispherical base?

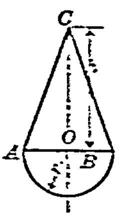
NOTE: The body is in stable equilibrium when its center of gravity does not lie above the plane AB.

Solution:

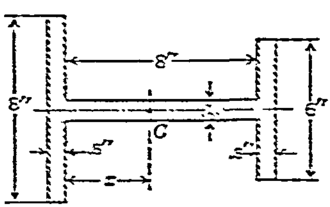
The limiting value of the distance $\bar{x}$ from the point of support to the centroid of the body is $\bar{x} = r$. Considering the body as consisting of two parts, the hemisphere and the cylinder (§§ 33, 35), we find

$$\bar{x}\cdot\left(\frac{2}{3}\pi r^2 + \pi r^2 h\right) = \left(\frac{2}{3}\pi r^2\right)\cdot\frac{5}{8}r + (\pi r^2 h)\cdot\left(\frac{h}{2} + r\right).$$

With $\bar{x} = r$, the equation gives $h = \frac{r}{2}\sqrt{2} = 0.707r$.



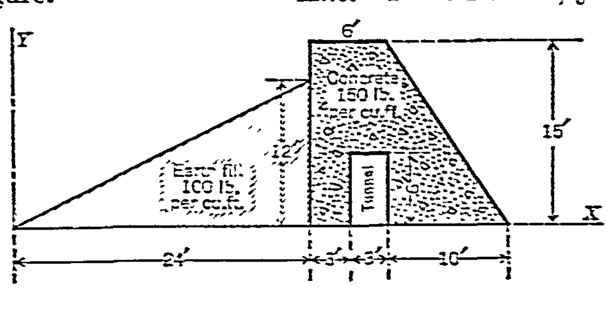
296. A body consisting of a cone and a hemisphere, as shown in the sketch, stands on its hemispherical base. Find the maximum altitude h of the cone for which the body will be in stable equilibrium in the position shown. *Ans.* $h = 1.73r$.

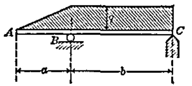


297. Find the centroid of the I-beam section, the dimensions of which are given in the sketch.

Ans. $\bar{x} = 3.6$ in.

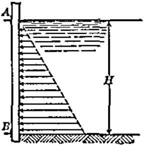
298. Find the center of gravity of the dam cross-section shown in this figure. *Ans.* $\bar{x} = 24.51$ ft.; $\bar{y} = 5.67$ ft.





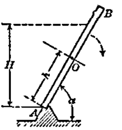
299. A horizontal beam AC supported at B and C carries between B and C a distributed load of intensity q lbs. per unit length; between B and A the load intenstiy decreases to zero, as shown in the sketch. Find the reactions at B and C, neglecting the weight of the beam.

Ans. $R_B = \frac{q}{6}\left(3a + 3l + \frac{a^2}{l}\right)$, up; $R_C = \frac{q}{6}\left(3l - \frac{a^2}{l}\right)$, up.



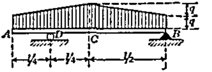
300. A vertical shield of a dam carries the pressure of salt water to a depth $H = 12$ ft. The shield is supported at A and B. A cubic foot of the water weighs $q = 64$ lbs. Find the linear reactions of the supports A and B.

Ans. $R_A = 1536$ lbs./ft.; $R_B = 3072$ lbs./ft.



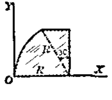
301. A rectangular gate AB of an irrigation canal is built as shown in the sketch. It can rotate about a pivot O. When the water is low, the gate is closed, but when the water reaches a level H, the gate swings about the pivot and opens the canal. Neglecting friction, find the height H above the lower edge A of the gate when it will open.

Ans. $H = 3h \sin \alpha$.



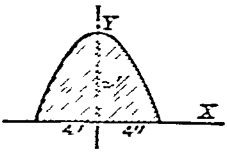
302. A beam AB carries a distributed load shown in the sketch. The intensity of the loading is q lbs. per unit of length at the ends A and B, and $2q$ lbs. per unit of length at the center of the beam. Find the reactions of the supports B and D.

Ans. $R_B = \frac{1}{2}ql$ lbs.; $R_D = ql$ lbs.



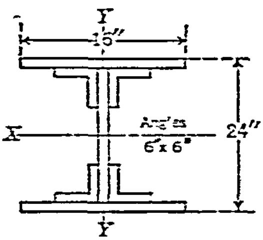
303. Find the coordinates of the centroid of the shaded area shown in sketch.

Ans. $\bar{x} = 0.56R$, $\bar{y} = 0.40R$.



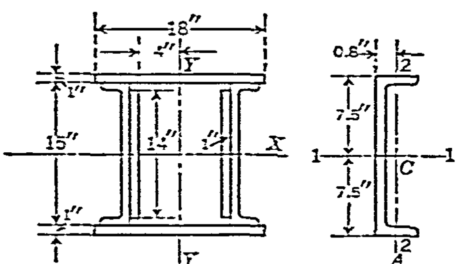
304. Find the centroid of the parabolic segment shown. *Ans.* $\bar{x} = 0$, $\bar{y} = 2.4$ in.

10. Moment and Product of Inertia of Plane Areas.

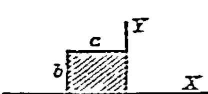


305. A built-up girder is made up of two 16 in. $\times$ 1 in. cover plates, one web plate 22 in. $\times$ 1 in., and four angles, each 6 in. $\times$ 6 in. $\times$ 1 in. Determine the moments of inertia with respect to the x and y axes shown.

Ans. $I_x = 8930$ in.^{4}; $I_y = 1072$ in.^{4}.

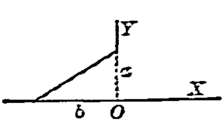


306. A built-up column is made up of two cover plates 18 in. $\times$ 1 in., two channels 15 in. $\times$ 35 lbs., and two web plates 14 in. $\times$ 1 in. The centroid of a single channel is at C, as indicated in Sketch A, its moment of inertia is 8.4 in.^{4} with respect to axis 2–2 and 318.7 in.^{4} about axis 1–1; the area of one channel is 10.23 sq. in. Determine the moments of inertia of the column section with respect to the x and y axes. *Ans.* $I_x = 3404$ in.^{4}; $I_y = 2248$ in.^{4}.



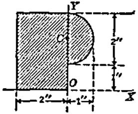
307. Determine the product of inertia of the shaded area with respect to the axes given. (Derive by direct integration.)

Ans. $P_{xy} = -\frac{1}{4}a^2b^2$.

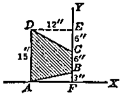


308. Determine the product of inertia of the shaded area with respect to the axes given. (Derive by direct integration.)

Ans. $-\dfrac{1}{24}a^2b^2$.



309. Compute the product of inertia of the shaded area with respect to the x and y axes. *Ans.* $P_{xy} = -7.67$ in.^{4}.



310. Determine the product of inertia, P_{xy}, for the shaded area shown.

Solution:

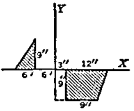
The product of inertia of the shaded area $ABCD$ is equal to the product of inertia of area $AFED$ minus the products of inertia of areas ABF and DCE (§ 42).

For $AFED$, $P_{xy} = 180\,(6)\,(-7.5) = -8100$ in.^{4}.

For ABF, $P_{xy} = \dfrac{3^2 \times \overline{12}^2}{72} + \dfrac{1}{2} \times 3 \times 12 \times (4)\,(-1) = -54$ in.^{4}.

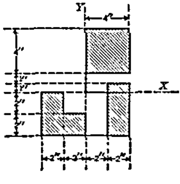
For DCE, $P_{xy} = -\dfrac{6^2 \times \overline{12}^2}{72} + \dfrac{1}{2} \times 6 \times 12 \times (4)\,(-13) = -1944$ in.^{4}.

$P_{xy} = -8100 - [-54 - 1944] = -6102$ *in.*^{4}.



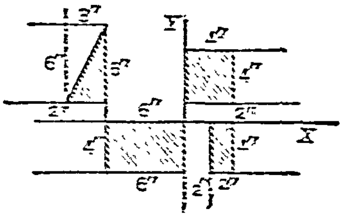
311. For the shaded area shown: (*a*) Locate the centroid. (*b*) Determine the product of inertia, P_{xy}. (*c*) Determine the product of inertia, $\overline{P}_{xy}$, for axes parallel to the given axes and passing through the centroid.

Ans. $\bar{x} = 4.67$ in.; $\bar{y} = -2.67$ in.; $P_{xy} = -3860$ in.^{4}; $\overline{P}_{xy} = -2346$ in.^{4}.



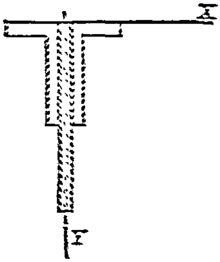
312. For the shaded area shown: (*a*) Locate the centroid. (*b*) Find the product of inertia, P_{xy}, for the axes shown. (*c*) Find the moment of inertia, $\bar{I}_x$, for an axis passing through the centroid.

Ans. $\bar{x} = 0.89$ in.; $\bar{y} = 0.552$ in.;
$P_{xy} = +143.0$ in.^{4};
$\bar{I}_x = 389$ in.^{4}.



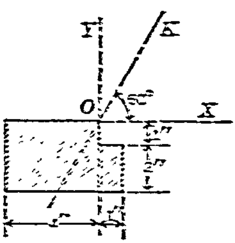
313. For the shaded area shown: (a) Locate the centroid. (b) Find the moment of inertia, I_y, about the y axis. (c) Find the product of inertia P_{xy}.

Ans. $\bar{x} = -1.39$ in.; $\bar{y} = 0.63$ in.;
$I_y = 894$ in.^{4};
$P_{xy} = -23.5$ in.^{4}.



314. A structural section is made up of one web plate 15 in. $\times$ 1 in., and two angles 8 in. $\times 4$ in. $\times$ 1 in., as shown in the sketch. (a) Locate the centroid. (b) Determine the principal axes and principal moments of inertia for axes passing through the centroid.

Ans. $\bar{x} = 0$; $\bar{y} = 4.85$ in.; $\bar{I}_x = 598$ in.^{4};
$\bar{I}_y = 77$ in.^{4}.



315. For the shaded area shown: (a) Compute I_x, I_y, and P_{xy}. (b) Find the moment of inertia about an axis OK making an angle of 60° with the x axis. (c) Determine the principal axes of inertia passing through point O, and the corresponding principal moments of inertia.

Solution:

The values are

(a) (§§ 40, 41, 42),

$$I_x = \frac{1}{3} \times 5 \times 3^3 - \frac{1}{3} \times 1 \times 1^3 = 44.7 \text{ in.}^4,$$

$$I_y = \frac{1}{3} \times 3 \times 4^3 + \frac{1}{3} \times 2 \times 1^3 = 64.7 \text{ in.}^4,$$

$$P_{xy} = 12 \times (+2)(+1.5) + 2\left(-\frac{1}{2}\right)(+2) = +34 \text{ in.}^4$$

(b) (§ 44),

$$I_{ok} = 44.7 \cos^2 60° + 64.7 \sin^2 60° - 2 \times 34 \sin 60° \cos 60°$$
$$= 11.2 + 48.5 - 29.4 = 30.3 \text{ in.}^4$$

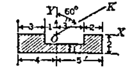
(c) (§ 45),

$$\tan 2\alpha = \frac{2 \times 34}{64\ 7 - 44\ 7} = 3\ 4$$

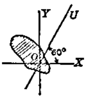
$$2\alpha = \begin{cases} 73°\ 30 \\ \text{or} \\ 253°\ 30 \end{cases} \qquad \alpha = \begin{cases} 36°\ 45 & \text{Axis of Min I,} \\ \\ 126°\ 45 & \text{Axis of Max I} \end{cases}$$

$$I_M = \frac{44\ 7 + 64\ 7}{2} \pm \sqrt{\left(\frac{64\ 7 - 44\ 7}{2}\right)^2 + \overline{34}^2} = 54\ 7 \pm 35\ 5,$$

$I_{max} = 90\ 2$ *in* ^{4}
$I_{min} = 19\ 2$ *in* ^{4}

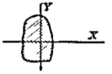


316 For the shaded area shown (a) Determine I_x, I_y, and P_{xy} (b) Determine the moment of inertia about axis OK making an angle of 30° with the x axis (c) Determine the principal axes of inertia passing through point O (d) Determine the principal moments of inertia

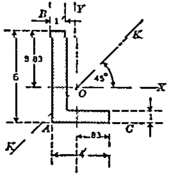


317. It is known that for the plane area shown $I_x = 400$ in ^{4}, $I_y = 150$ in ^{4}, $P_{xy} = -200$ in ^{4} Determine

(a) The directions of the principal axes (b) The values of the principal moments of inertia (c) The moment of inertia of the area with reference to the axis O–U, inclined at an angle of 60° to the x axis

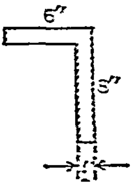


318 For a given area, $I_x = 120$ in ^{4}, $I_y = 60$ in ^{4}, $P_{xy} = -40$ in ^{4} (a) Determine the principal moments of inertia (b) Determine the moment of inertia about an axis inclined at an angle of 30° to the x axis



319 For the shaded area shown in this figure, $I_x = 30\ 75$ in ^{4}, $I_y = 10\ 75$ in ^{4}, $\overline{P}_{xy} = -10\ 00$ in ^{4} The x and y axes pass through the centroid Determine the moment of inertia with respect to the axis AB Determine the product of inertia with respect to the axes AB and AC Determine the principal moments of inertia for axes through the centroid O Determine the moment

of inertia with respect to an axis KK which passes through the point O and is inclined at 45° to the x axis.

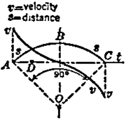


320. For the structural angle shown: (*a*) Locate the centroid. (*b*) Determine the principal axes of inertia for axes passing through the centroid. Compute the corresponding principal moments of inertia.

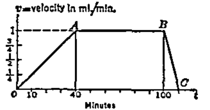
PART II. KINEMATICS

MOTION OF A POINT

11. Rectilinear Motion of a Point.



321. The space-time curve for a certain motion is the quarter circle shown in the sketch. Draw the velocity-time curve.



322. The broken line $OABC$ and the Ot axis beyond the point C form a diagram of train speeds in miles per minute. Find the distance from the starting point that the train traveled, as a function of time during the periods: (1) from $t = 0$ to $t = 40$ min.,
(2) from $t = 40$ to $t = 100$ min.,
(3) from $t = 100$ to $t = 110$ min.,
(4) from $t = 110$ to $t = 120$ min.

Ans. (1) $S = 0.0125\,t^2$ mi.; (2) $S = (t - 20)$ mi.;
(3) $S = (11t - 0.05t^2 - 520)$ mi.; (4) $S = 85$ mi.;
(t measured in minutes).

323. A point travels on a straight line. Its distance in inches from a fixed point on the line is $s = 4\,t - 2t^2$. Find the velocity v and the acceleration a of the point at any time t. Draw the space-time and velocity-time curves.

Ans. $v = (4 - 4t)$ in./sec.; $a = -4$ in./sec.^{2}.

324. A point moves in a straight line in accordance with the law $s = (t^3 - 40t)$ feet. Find the velocity when $t = 5$ sec. What is the average velocity for the second preceding the instant named? Where and when does the particle stop?

Ans. $v_5 = 35$ ft./sec.; $v_{av} = 21$ ft./sec.
Stop at $t = 3.65$ sec.; $s = -97.4$ ft.

325. A point moves in a straight line in accordance with the law $s = 36t - 3t^3$, where s is in feet and t is in seconds. Find the velocity when $t = 3$ sec. and when $t = 8$ sec. What is the average velocity for the second preceding the instants named? For the second following the instants? Does the particle stop? If so, where? When?

Solution:

The velocity of the point is (§§ 50, 51), $v = \dfrac{ds}{dt} = -9t^2 + 36,$

$$t = 3, \qquad v_3 = -9\,(3)^2 + 36 = -45 \textit{ ft./sec.},$$
$$t = 8, \qquad v_8 = -9\,(8)^2 + 36 = -540 \textit{ ft./sec.}$$

The average velocity is

$$v_{av} = \frac{\text{Displacement during time interval}}{\text{Time interval}},$$

$$v_{2-3} = \frac{s_3 - s_2}{t_3 - t_2} = \frac{+27 - 48}{3 - 2} = -21 \textit{ ft./sec.},$$

$$v_{3-4} = \frac{s_4 - s_3}{t_4 - t_3} = \frac{-48 - 27}{4 - 3} = -75 \textit{ ft./sec.},$$

$$v_{7-8} = \frac{s_8 - s_7}{t_8 - t_7} = \frac{-1248 - (-777)}{8 - 7} = -471 \textit{ ft./sec.},$$

$$v_{8-9} = \frac{-1863 - (-1248)}{9 - 8} = -615 \textit{ ft./sec.}$$

The particle stops when $v = 0$.

$$v = -9t^2 + 36 = 0, \qquad t = \pm 2 \text{ sec.}$$

Using only $t = +2$ sec., the particle stops at

$$s_{t=2} = -3\,(2)^3 + 36\,(2) = +48 \textit{ ft.}$$

326. A ship, while being launched, slipped down the skids with a uniform acceleration. The first foot was traversed in 10 seconds. How long did it take to pass over the skids? The length of the skids was 400 ft. *Ans.* $T = 3$ min., 20 sec.

327. A shell leaves the muzzle of a gun with a velocity of 1500 ft./sec. Assuming a uniform acceleration during the motion of the shell inside the gun, find the time it took to travel through the gun barrel, which is 3 ft. long. *Ans.* $T = 0.004$ sec.

328. A train leaves a station with a uniform acceleration of ⅓ ft./sec.². At what distance from the station will its speed be 48 mi./hr.? *Ans.* $S = 7460$ ft.

329 A train moves with a velocity of 48 mi /hr The brakes can retard the train at the rate of 1 2 ft /sec ^{2} How far from a station should the brakes be applied?

Ans $S = 2070$ ft $T = 59$ sec

330 The ram of a pile driver hits a pile and travels with it after the impact The pile is driven in 3 inches, moving this distance in 0 02 seconds If the motion were uniformly decelerated what was the velocity of the ram at the instant of impact?

Ans 25 ft /sec

331 Water drips from a pipe at the uniform rate of 10 drops per second After a drop has fallen for one second, what is the distance between it and the drop following it? *Ans* 3 06 ft

332 A point starting from rest moves on a straight line with an acceleration of 12 ft /sec ^{2} Another point starts from the same place as the first point two seconds later, and moves with a uniform velocity of 54 ft /sec in the same direction How soon will the second point reach the first?

Ans One second after it starts

333 Solve the previous problem with the additional condition that the first point starts with an initial velocity of 12 ft /sec

Ans The points will not meet

334 The acceleration of a point is $12t$ in /sec ^{2}, directed along the x axis in a negative direction At $t = 2$ sec its velocity $v = 6$ in /sec is directed along the x axis in the positive direction When $t = 3$ sec, the point is 50 in from the origin Find the equation of motion

Solution

Calling the distance of the point from the origin x we find (§ 57) that the equation of motion is

$$\frac{d^2x}{dt^2} = -12t \qquad \text{when } t = 2 \quad \frac{dx}{dt} = 6 \qquad \text{when } t = 3 \quad x = 50$$

Integrating we have

$$\frac{dx}{dt} = -6t^2 + C_1$$
$$x = -2t^3 + C_1t + C_2 \qquad C_1 = +30 \qquad C_2 = +14$$
$$x = -2t^3 + 30t + 14$$

335 A point moves on a straight line Its motion is described by the equation $t = c\log_{10}(b + s)$, where s is the distance of the

point from a fixed reference point and c and b are constants. Find the velocity v and the acceleration a of the point at any time t.

Ans. $v = \left(\frac{1}{c}\log_e 10\right) \times 10^{t/c}$; $\quad a = \left(\frac{1}{c}\log_e 10\right)^2 \times 10^{t/c}$.

336. The motion of a point moving on a straight line is described by the equation

$$x = \frac{mv_0}{k}(1 - e^{-kt/m}),$$

where v_0, m, k, and e are constants. Describe the character of this motion in physical terms. Find the acceleration a as a function of the velocity v.

Ans. $a = -\frac{k}{m}v$.

337. A point moves along a straight line, the distance from a fixed point being $s = a \sin kt$, where $a = 4$ in. and $k = \frac{1}{2}$ rad./sec. Draw the curves of position, velocity, and acceleration as functions of the time. *Ans.* $s = (4 \sin \frac{1}{2}t)$ in.; $v = (2 \cos \frac{1}{2}t)$ in./sec.; $a = (-\sin \frac{1}{2}t)$ in./sec.^{2}.

338. A point moves in a straight line in accordance with the law $s = 2 \sin (0.05t + 2)$, where s is in inches, t in seconds and the angle in radians. Determine the velocity and acceleration when $t = 10$ sec. and when $t = 75$ sec. Interpret the signs of your results.

Ans. (1) $v = -0.080$ in./sec.; $a = -0.0030$ in./sec.^{2};
(2) $v = 0.086$ in./sec.; $a = 0.0025$ in./sec.^{2}.

339. The acceleration of a particle moving along a straight line is expressed by $a = -32 \sin (4t + 30°)$ in./sec.^{2}. What is the amplitude? What is the frequency? What is the period? What is the angle of lead? Give the equation connecting v and t, and that connecting s and t.

Solution:

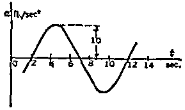
The equations of motion are (§ 57):

$$\begin{aligned} a &= -32 \sin (4t + 30°), \\ v &= \int a\,dt = 8 \cos (4t + 30°) + C_1, \\ s &= \int v\,dt = 2 \sin (4t + 30°) + C_1 t + C_2. \end{aligned}$$

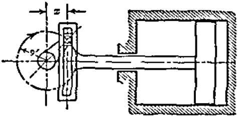
Assuming $C_1 = C_2 = 0$, the resulting equation is one of simple harmonic motion.

Amplitude = *2 inches*, Frequency $= \frac{4}{2\pi} = \frac{2}{\pi}$ *cycles per second*,

Period $= \frac{2\pi}{4} = \frac{\pi}{2}$ *seconds*, Angle of lead $= 30° = \pi/6$.

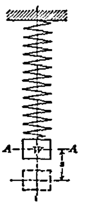


340. From this a–t curve of a simple harmonic motion determine the frequency, the amplitude and the angle of lead or lag. Write out the a–t, v–t, and s–t equations, and plot the curves of the last two.



341. This apparatus is being used to compress air. The crank is turning clockwise at 150 r.p.m. The stroke is 18 inches. Determine the acceleration of the piston when $x = 3$ in.

Ans. 739 in./sec.².



342. The body W is supported by a helical spring. The block is pulled down a distance of 3 inches, and is then released, from rest. It then executes a simple harmonic motion through AA as the central position with an up and down deflection of 3 inches. The stiffness of the spring and weight of the block are such that the acceleration of the block W is given by the law $d^2s/dt^2 = a = -72s$, in which the acceleration is expressed in feet per sec. per sec. and s in feet. Calculate the maximum velocity and acceleration of the block and period of vibration.

Ans. $v_{max} = 2.12$ ft./sec.; $a_{max} = 18$ ft./sec.²; $T = 0.74$ sec.

12. Curvilinear Motion of a Point.

343. A point moves counterclockwise on the circumference of a circle whose radius is 4 ft., starting at the right extremity of the horizontal diameter and moving with a constant speed of one revolution in 3 seconds. When the point has covered an arc of

150 degrees from the starting point, what are the axial components of its velocity and acceleration, if the horizontal and vertical diameters are the coordinate axes?

Ans. $v_x = -4.19$ ft./sec.; $v_y = -7.27$ ft./sec.; $a_x = 15.2$ ft./sec.^{2}; $a_y = -8.8$ ft./sec.^{2}.

344. The motion of a point is given by the equations $x = 3t$ in.; $y = 4\cos 4\pi t$ in. Find the equation of, and plot the path of the point.

Ans. $y = 4\cos 4\pi \frac{x}{3}$.

345. The motion of a point is given by means of the equations $x = 10\cos\left(2\pi\frac{t}{5}\right)$ in.; $y = 10\sin\left(2\pi\frac{t}{5}\right)$ in. Find the path of the point, the magnitude and direction of its velocity v, and the magnitude and direction of its acceleration a.

Ans. The path is $x^2 + y^2 = 100$ in.^{2}; $v = 12.56$ in./sec.; $a = 15.7$ in./sec.^{2}.

346. A point moves with a constant velocity of 3 in./sec. directed at an angle of $\frac{\pi}{2}t$ radians to the x axis. At the time $t = 0$ the point was at the origin O of the coordinate system. Find the equation of the path of motion.

Solution:

The components of velocity of the point are:

$$v_x = \frac{dx}{dt} = 3\cos\frac{\pi}{2}t, \qquad v_y = \frac{dy}{dt} = 3\sin\frac{\pi}{2}t.$$

Integrating (§ 57*a*), with $x_0 = 0$, $y_0 = 0$, we have

$$x = \frac{6}{\pi}\sin\frac{\pi}{2}t, \qquad y = \frac{6}{\pi} - \frac{6}{\pi}\cos\frac{\pi}{2}t.$$

Eliminating the time (§ 48), we find that the path of the point is

$$\frac{6}{\pi}\sin\frac{\pi}{2}t = x, \qquad \frac{6}{\pi}\cos\frac{\pi}{2}t = \left(\frac{6}{\pi} - y\right); \qquad \text{or} \qquad x^2 + \left(y - \frac{6}{\pi}\right)^2 = \frac{36}{\pi^2},$$

which is a circle with the center at $\left(0, \frac{6}{\pi}\right)$.

347. A train leaves a station and moves with uniformly increasing velocity. In 2 minutes it reaches a speed of 36 miles per hour. The track is curved and has a radius of 1408 ft. Find the tangential, normal, and absolute accelerations of the train 1 min. and 20 sec. after it leaves the station.

Ans. $a_t = 0.441$ ft./sec.^{2}; $a_n = 0.880$ ft./sec.^{2}; $a = 0.984$ ft./sec.^{2}.

348 A shell leaves the muzzle of a gun with a velocity of 1500 ft per sec The gun is elevated at an angle of 30° to the horizontal Neglecting the effect of air resistance, find the radius of curvature ρ of the shell's path at its highest point

Solution

The shell moves with a horizontal acceleration $a_x = 0$ and a vertical acceleration $a_y = -g = -32\ 2$ ft /sec² $a = 32\ 2$ ft /sec² at any point of the shell s path

$$\frac{d^2x}{dt^2} = 0 \qquad v_x = 1500 \cos 30^\circ = 1299 \text{ ft /sec}$$

$$\frac{d^2y}{dt^2} = -32\ 2 \qquad v_y = 1500 \sin 30^\circ - 32\ 2t = 750 - 32\ 2t$$

At the highest point $v_y = 0$ the velocity of the shell is

$$v = v_x = 1299 \text{ ft /sec}$$

The acceleration at this point can be written (§ 55) $a = \sqrt{a_t^2 + a_n^2}$ but at the highest point, $a_t = a_x = 0$ and $a = a_n = \dfrac{v^2}{\rho} = 32\ 2$ ft /sec² Therefore

$$\rho = \frac{v^2}{32\ 2} = \frac{(1299)^2}{32.2} = 52\ 400 \text{ ft}$$

349 The motion of a point is given by the equations $x = at$ and $y = bt - gt^2/2$ Find the tangential and normal accelerations of the point

$$Ans \quad a_t = -g\frac{b - gt}{\sqrt{a^2 + (b - gt)^2}}, \quad a_n = -g\frac{a}{\sqrt{a^2 + (b - gt)^2}}$$

350 A point has a helical motion defined by the equations $x = 2 \cos 4t$, $y = 2 \sin 4t$, and $z = 2t$ x, y, and z are expressed in feet Find the radius of curvature of the path of the point

Ans $\rho = 2\frac{1}{8}$ ft

351 Three bullets, shot horizontally from three points on the bank of a lake at heights h_1, h_2, and h_3 above the surface of the lake, leave with initial velocities of 150, 225, and 300 ft /sec and all strike the water at the same time The first bullet, which travels the least distance, strikes the water 300 ft from the shore Find the time T the bullets are in the air, and the velocities v_1 v_2 and v_3 at the instant they hit the water Neglect the effects of air friction

Ans $T = 2$ sec, $v_1 = 163$ ft /sec, $v_2 = 234$ ft /sec, $v_3 = 307$ ft /sec

352. The motion of a point is given by the equations

$$x = a\cos(\alpha + \omega t), \qquad y = b\sin(\beta + \omega t),$$

where a, b, α, β, and ω are constants. Find the equation of the path over which the point travels.

Ans. An ellipse, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + 2\dfrac{xy}{ab}\sin(\alpha - \beta) = \cos^2(\alpha - \beta)$.

353. The motion of a point is given by the equations

$$x = v_0 t\cos\alpha, \qquad y = v_0 t\sin\alpha - \tfrac{1}{2}gt^2.$$

Find:

(1) the path of the point;
(2) the coordinates of the highest point of the path;
(3) the projections of the velocity at the moment when the point crosses the x axis. Explain the kinematic meaning of v_0 and α.

354. The motion of a point is determined by the equations of the previous problem: $v_0 = 60$ ft./sec., $\alpha = 60°$, $g = 32.2$ ft./sec.^{2}. At the moment $t = 0$, another point starts from the origin O and moves uniformly along the axis OX. What should be the velocity v_1 of the second point in order that the points meet? Find the coordinate x_2 of the meeting point.

Ans. $x_2 = \dfrac{v_0^2\sin 2\alpha}{g}$; $v_1 = v_0\cos\alpha$.

355. A particle moves with uniform velocity in guides along the equator of the earth. The radius of the earth at the equator is 637×10^5 cm. and the acceleration of gravity is $g = 978$ cm./sec.^{2}. At what velocity must the particle move to reach an acceleration equal to g? How long would it take the particle to go completely around the earth at this velocity?

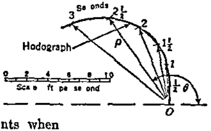
Ans. $v = 7.9$ km./sec.; $T = 1.41$ hr.

356. A point moves counterclockwise on the circumference of a circle whose radius is 20 ft., starting at the right extremity of a horizontal diameter and traversing distance s, so that $s = 2t^2$, where t is the time in seconds after starting and s is in feet. Using half-second intervals, draw the hodograph for the first three seconds. Then determine the magnitude and direction of the acceleration when $t = 3$ sec.

Solution

The radii vectors ρ of the hodograph represent the velocities of the point while the directional angles θ are the angles between the velocities and the horizontal

$$\rho = \frac{ds}{dt} = 4t \text{ ft /sec}, \qquad \theta = \frac{\pi}{2} + \frac{s}{20} = \left(\frac{\pi}{2} + 0\ 1t^2\right) \text{rad}$$



At the specified instants when

$t =$	0	½	1	1½	2	2½	3	sec
$\rho =$	0	2	4	6	8	10	12	*ft /sec*
$\theta =$	1 571	1 596	1 671	1 796	1 971	2 196	2 471	rad
$\theta =$	90°	91°30	95°30	102°50	113°0	125°50	141°30	

The acceleration a of the point is equal to the velocity u of the hodograph point (§ 59) $a = u = \sqrt{u_\rho^2 + u_\theta^2}$ where u_ρ and u_θ are (§ 56) the components of the hodograph point along and normal to the radius vector

$$u_\rho = \frac{d\rho}{dt} = 4 \text{ ft /sec}^2, \qquad u_\theta = \rho\frac{d\theta}{dt} = 4t \times 0\ 2t = 0\ 8t^2 = 7\ 2 \text{ ft /sec}^2$$

$$a = u = \mathit{8.23} \text{ ft /sec}^2$$

(The acceleration may be found also (§ 55)

$$a = \sqrt{a_n^2 + a_t^2}, \qquad a_t = \frac{d^2s}{dt^2} = 4 \text{ ft /sec}^2$$

$$a_n = \frac{t^2}{R} = \frac{12^2}{20} = 7\ 2 \qquad a = \mathit{8.23\ ft\ /sec^2}$$

357 A point moves clockwise on the circumference of a circle whose radius is 30 ft, starting at the right extremity of a horizontal diameter and traversing distance s so that $s = 3t^3$, where t is the time after starting in seconds and s is in feet Using half-second intervals, draw the hodograph for the first three seconds Then determine the magnitude and direction of the acceleration when $t = 2\ 5$ *Ans* $a = 114\ 7$ *ft /sec*^{2}, $\theta_a = 66\ 5°$

358 A point starts at time $t = 0$ *from a point* (1, 2, 4) and moves with uniform velocity $v = 24$ ft /sec along a line which has the direction cosines ⅓ ⅔, cos γ Find the equation of the path of the point and the hodograph of its velocity

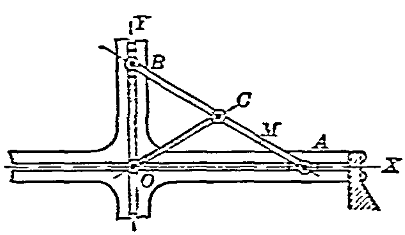
Ans $2x = y = z - 2$ The hodograph is a point $x_1 = 8,\ y_1 = 16,\ z_1 = 16$

359. A shell leaves the muzzle of a gun which is inclined at an angle of 30° to the horizontal. The muzzle velocity of the shell is 1500 ft./sec. Neglecting the effects of air resistance, find the hodograph of the velocity of the shell and the velocity v_1 of the point which traces the hodograph.

Ans. A vertical straight line, $x_1 = 750\sqrt{3}$ units; $v_1 = -32.2$ units/sec.

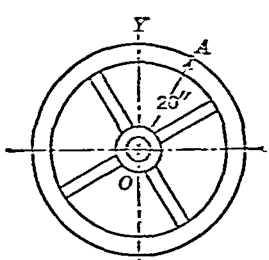
360. A body rotates at the uniform speed of 30 r.p.m. Find the hodograph of the velocity of a point on the body, located at a distance of 2 ft. from the axis of rotation, and the velocity v_1 of the point tracing the hodograph.

Ans. A circle of radius 2π units; $v_1 = 2\pi^2$ units/sec.



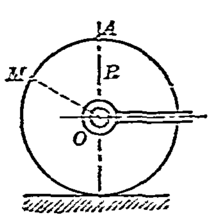
361. The sliding bar AB of an ellipsograph is 20 in. long. The crank OC is 10 in. long and $AC = BC$. The crank rotates around O with a constant angular velocity ω. A pencil is attached to the sliding bar at M, 5 in. from A. Find the equation of the curve traced by the pencil and the equation of the hodograph of the pencil-point velocity.

Ans. $\dfrac{x^2}{225} + \dfrac{y^2}{25} = 1$; the hodograph is $\dfrac{x_1^2}{225} + \dfrac{y_1^2}{25} = \omega^2$.



362. A flywheel starts from rest rotating with uniform acceleration. In 22 seconds it reaches a speed of 105 r.p.m. Point A on the flywheel is 20 inches from the center. At the time the flywheel begins to rotate it is on the vertical line through the center. Find the equation of the hodograph of the velocity of A.

Ans. $\rho = 20\sqrt{\theta}$ units.



363. A locomotive runs at a speed $v_0 = 72$ miles per hour. The driving wheels are 80 inches in diameter and roll without slipping on the rail. Find the value and direction of the velocity v of a point M on the rim of the wheel. Find the equation of the hodograph of velocity

absolute linear acceleration a of a point on the surface of the shaft, at any time t. *Ans.* $\omega = 20t$ rad /sec ; $\alpha = 20$ rad /sec ^{2}, $a = 80\sqrt{1 + 400t^4}$ in /sec.^{2}.

375. A flywheel of 12 ft. diameter rotates with a uniform retardation. It made 600 revolutions from $t = 0$ to $t = 20$ sec. At the time $t = 15$ sec. its angular velocity was $\omega_1 = 30\pi$ rad /sec. Find the acceleration of a point on the rim at the time $t = 20$ sec.

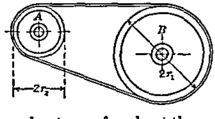
Solution

The equation of motion of the fly wheel is (§ 63) $\frac{d^2\theta}{dt^2} = \alpha$, a constant, at the time $t = 0$, $\theta = 0$, and at $t = 15$, $\frac{d\theta}{dt} = 30\pi$ rad /sec Integrating, we find

$$\omega = \omega_0 + \alpha t, \qquad \theta = \omega_0 t + \frac{1}{2}\alpha t^2 + \theta_0,$$

$$\alpha = -6\pi \text{ rad /sec}^2, \qquad \omega_0 = 120\pi \text{ rad /sec}$$

At $t = 20$ sec, $\omega = 0$, $a_p = r\alpha = 6 \times 6\pi = 113.2$ *ft /sec* ^{2}.



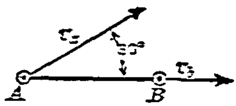
376. A generator with a pulley A is driven by means of a belt from a pulley B on a prime mover. The radius of B is $r_1 = 30$ in and the radius of A is $r_2 = 12$ in The prime mover starts from rest and accelerates uniformly at the rate of 0.4π rad /sec ^{2}. Find the time necessary to bring the generator up to a speed of 300 r p m Assume that the belt does not slip. *Ans.* 10 sec.

377. A body oscillates around a fixed axis. Its angular position at any time t is described by the equation $\phi = 20° \sin \frac{t}{5} 10°$, where t is time in seconds. Find:

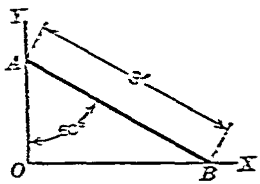
(1) The angular velocity ω of the body at the time $t = 0$.
(2) The times t_1 and t_2 at which the direction of rotation changes in the first cycle
(3) The duration T for one complete cycle

Ans. (1) $\omega = 0.0123$ rad /sec.; (2) $t_1 = 45$ sec ; $t_2 = 135$ sec ; (3) $T = 3$ min.

14. Motion of a Rigid Body Parallel to a Fixed Plane.



378. A rod AB, 30 in. long, moves in the plane of the drawing. At a certain moment A is moving in a direction at 30° to the line AB with a velocity of $v_a = 180$ in./sec. while point B is moving in the direction of the line AB. Find the velocity v_b of point B at this moment. *Ans.* $v_b = 156$ in./sec.

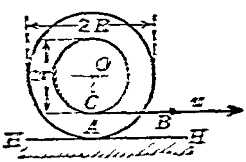


379. The two ends of a rod AB, 3 ft. long, slide along mutually perpendicular lines OX and OY. Find the coordinates x and y of the instantaneous center of rotation when angle $OAB = 60°$.

Solution:

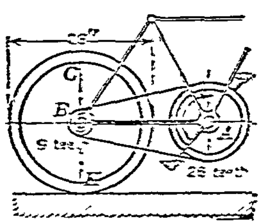
The instantaneous center is (§ 69) at the intersection of the perpendicular to OY at A and the perpendicular to OX at B:

$$x = OB = 2.60 \text{ ft.}; \qquad y = OA = 1.50 \text{ ft.}$$

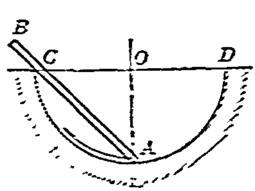


380. A spool lies on a horizontal plane HH. The radius of the flange of the spool is R and the radius of the cylinder is r. A thread AB wound around the cylinder is pulled horizontally with a velocity of u; the spool rolls without sliding. Find the velocity v of the center O of the spool.

$$\textit{Ans.} \quad v = u\frac{R}{R-r}.$$



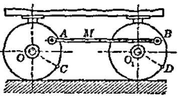
381. The pedal sprocket A of a bicycle has 26 teeth. The wheel sprocket B has 9 teeth. The wheel C has a diameter of 28 in. Find the velocity of the bicycle when the pedals are turned at one revolution per second. *Ans.* 14.4 mi./hr.



382. A straight rod AB moves in the plane of the sketch. The end A moves on the surface of a cylinder CAD and the side of the rod slides on the point C. At the instant the radius OA is perpendicular to the diameter CD, the point A has a velocity

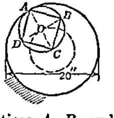
of 4 ft./sec. Find the velocity v_c of the point touching C at this moment.

Ans. $v_c = 2.83$ ft./sec.



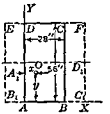
383. The crank pins A and B of the locomotive driving wheels O and O_1 are connected by a side rod, the length of which is equal to the center distance OO_1. The wheels are 4 ft. in diameter and $OA = O_1B = 1$ ft. Find the absolute acceleration of any point M on the side rod when the train is moving at a speed of 36 miles per hour.

Ans. 697 ft./sec.^{2}.



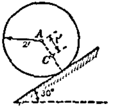
384. A circle of 10 in. diameter rolls on the inside of the circumference of another circle of 20 in. diameter. The center of $ABCD$ moves on a circle at the uniform velocity of one revolution per second. Draw the space-centrode and the body-centrode. Find the velocities of the vertices A, B, and C of a square inscribed in the smaller circle at the instant when A is in contact with the larger circle.

Ans. $v_a = 0$; $v_b = 44.4$ in./sec.; $v_c = 62.8$ in./sec.



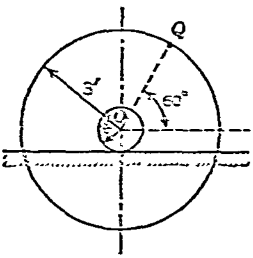
385. The top $ABCD$ of a folding table is rectangular in shape. $AB = 28$ in. and $AD = 56$ in. In order to unfold the table, the top is rotated 90° around the pin O until it is in the position $A_1B_1C_1D_1$; where $AB_1 = BC_1$. The table can then be unfolded to have the square top B_1EFC_1. Find the position of the pin.

Ans. $x = 7$ in.; $y = 21$ in.



386. A disc, 4 ft. in diameter, shown in its initial position in the sketch, starts from rest and rolls without slipping down a 30° inclined plane. Its angular velocity increases at the rate of 5 radians per sec. per sec. Determine the position, the absolute velocity, and the absolute acceleration of point C at the instant $t = 4$ sec.

Ans. C is 132° from its original position; $v_c = 55.3$ ft./sec.; $a_c = 408$ ft./sec.^{2}.

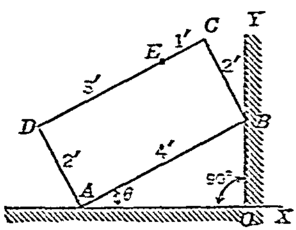


387. A turbine disc 6′ in diameter, mounted on a shaft 16″ in diameter, is rolled on horizontal parallels. When the point Q is in the position shown, the speed of rotation of the rolling disc is two revolutions per second in the clockwise direction. The rotation is being retarded at the rate of one revolution per second. Find the velocity and acceleration of point Q.

Ans. $v_q = 542$ in./sec.; $a_q = 1827\pi$ in./sec.^{2}.

388. A and B are two points in a body which are 8 ft. apart. A moves up and down, while B moves to the right and left along a horizontal line. When the line AB makes an angle of 30° with the horizontal, A is moving upward with a velocity of 8 ft./sec. and a deceleration of 12 ft./sec.^{2}. Determine the velocity and acceleration of a point P which is 2 ft. from A and on the line AB, between A and B. Determine the velocity and acceleration of a point Q which is on the line AB (extended), and is 2 ft. from A and 10 ft. from B.

Ans. $v_p = 6.11$ ft./sec.; $a_p = 9.1$ ft./sec.^{2}; $v_q = 10.09$ ft./sec.; $a_q = 15.08$ ft./sec.^{2}.



389. The block $ABCD$ moves in such a way that the point B traverses a vertical line on the wall and A moves along the horizontal line at right angles to the wall. When $\theta = 30°$, the point A is moving with a velocity of 4 ft. per sec. toward the left and A has an acceleration of 24 ft./sec.^{2} towards the right. For this position of the rectangular block, find the magnitude and direction of the velocity for point E. Also find the components of the acceleration of E. Point E is on CD, 1 ft. from C.

Solution:

Both the velocity and acceleration of point E can be found by using point A as a base point. The angular velocity and acceleration are determined by using the given data concerning the motion of point A (§§ 66, 68). From Fig. a:

$$x = 4\cos\theta, \qquad \frac{dx}{dt} = -4\omega\sin\theta,$$

when $\theta = 30°$, $\frac{dx}{dt} = +4$, $+4 = -4\omega \times \frac{1}{2}$, $\omega = -2$ rad./sec. (clockwise),

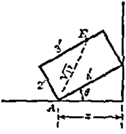
$$\frac{d^2x}{dt^2} = -4\alpha \sin\theta - 4\omega^2 \cos\theta, \quad \text{when} \quad \theta = 30°, \quad \frac{d^2x}{dt^2} = -24,$$

$$-24 = -4\alpha \cdot \frac{1}{2} - 4(-2)^2 \cdot (0.866) = +5.06 \text{ rad./sec.}^2$$

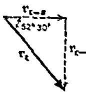
(counterclockwise)

The velocity of E (Fig. b) is given by

$$v_E = v_A \nrightarrow v_{E/A},$$
$$v_{E-x} = -4 + 7.21 \sin 63° 42' = +2.46 \text{ ft./sec.},$$
$$v_{E-y} = -7.21 \cos 63° 42' = -3.2 \text{ ft./sec.},$$
$$v_E = \sqrt{\overline{2.46}^2 + \overline{3.2}^2} = 4.04 \text{ ft./sec.},$$
$$\theta_y = 52° 30'.$$



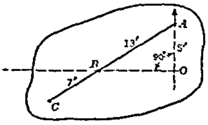
(a)



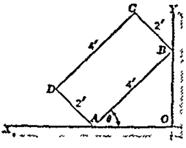
(b)

The acceleration components of E are:

$$a_E = a_A \nrightarrow (a_{E/A})_t \nrightarrow (a_{E/A})_n,$$
$$a_{E-x} = +1.8 \text{ ft./sec.}^2,$$
$$a_{E-y} = -4.8 \text{ ft./sec.}^2$$

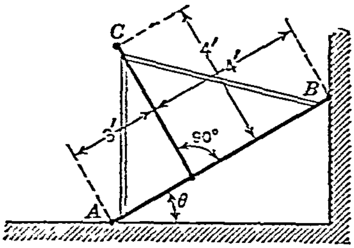


390. The body shown in outline has a plane motion so that B moves along the line BO while A moves along OA. $AB = 13$ feet. When A is 5 ft. from O and its velocity is 60 ft. per min. in the sense OA, what is the velocity of C, a point on the line AB and 7 feet from B? *Ans.* $v_C = 50.3$ ft./min.



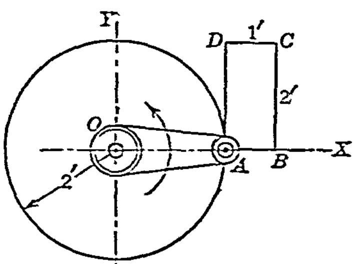
391. A rectangular block $ABCD$ moves so that the point B traverses a vertical line on the wall, and A moves along a horizontal line at right angles to the wall. The angle θ is given by the equation $\theta = -0.5t^2 + 0.5t + \pi/4 + 1$; where θ is in radians and t in seconds. Calculate

the velocity and acceleration of point C at the instant $t = 2$ sec. *Ans.* $v_C = 3.00$ ft./sec.; $a_C = 11.87$ ft. sec.^{2}.



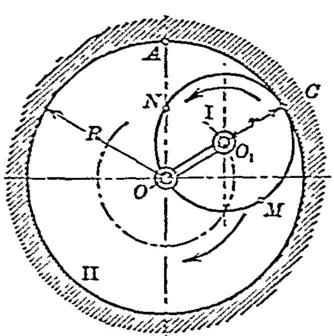
392. The point A on the frame ABC moves along a horizontal line and B along a vertical line. At a time when $\theta = 30°$, point A has a velocity of 7 ft./sec. toward the left and an acceleration of 34.75 ft./sec.^{2} toward the right. For this instant (a) find the velocity of point C. (b) Find the acceleration of point C.

Ans. $v_C = 3.16$ ft./sec.; $a_C = 25.15$ ft./sec.^{2}.



393. A rectangular plate $ABCD$ is mounted on a pivot at A, carried by the arm AO. The mechanism starts from rest in the position shown. The arm rotates counterclockwise, the motion of A being $s = \frac{1}{3}t^3$, where s is measured in feet of arc. At the same time the plate rotates clockwise about pivot A according to the law $\theta = 2t^2$, where θ is measured in radians. Give the position, velocity and acceleration of corner D at the instant $t = 2$ sec.

Ans. $x_D = 2.45$ ft.; $y_D = 1.66$ ft.; $v_D = 16.3$ ft./sec.; $a_D = 133.6$ ft./sec.^{2}.



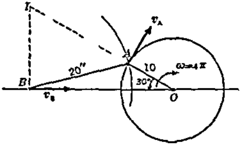
394. A disc I of radius r rolls in a clockwise direction on the inner surface of a fixed cylinder II of radius $R = 2r$. The axis O_1 makes a complete revolution in $\frac{1}{2}$ sec., and at time $t = 0$ it is on the vertical line OA. Find the path of any point M on the circumference of the disc. The point N is the intersection of the circumference of the disc and the diameter OA. Find the projection v_1 of its velocity on the line joining O and O_1. *Ans.* (1) Diameter of II passing through M; (2) $v_1 = -2\pi R \sin 2(\angle AOC)$.

395 The length of the crank of a reciprocating system is $OA = 10$ in, the connecting rod length is $AB = 20$ in. The crank rotates at a uniform speed of 2 revolutions per sec. Find the velocity of the cross head B when the angle $AOB = 30°$.

Solution

I is the instantaneous center of AB (§ 69). Therefore

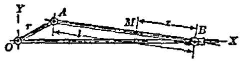
$$v_B = v_A \times \frac{IB}{IA}, \qquad \text{where} \qquad v_A = 10 \times 4\pi \text{ in/sec.}$$



From geometrical considerations

$$v_B = 4\pi(5 + \sqrt{5}) \text{ in/sec} = 91 \text{ in/sec}$$

396 A connecting rod AB of length l is attached to the end of a crank OA of length r, where r is small compared to l. The crank rotates at a constant angular velocity ω. Write approximate expressions for the x and y components of velocity and acceleration of a point M on the connecting rod at a distance z from B.



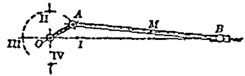
Ans

$$v_x = -\omega\left(r \sin\phi + \frac{l-z}{2}\frac{r^2}{l^2}\sin 2\phi\right),$$

$$v_y = \frac{zr}{l}\omega\cos\phi,$$

$$a_x = -\omega^2\left(r\cos\phi + \frac{(l-z)r^2}{l^2}\cos 2\phi\right),$$

$$a_y = -\frac{zr}{l}\omega\sin\phi$$

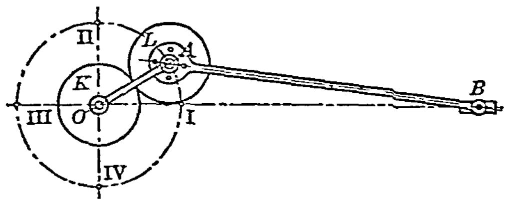


397 A connecting rod AB, 100 in long is attached to a crank OA, 20 in long. The crank rotates at a speed of 180 r p m. Find the angular velocity of the connecting rod and the linear velocity of its middle

point M in the four positions, when angle AOB is 0, $\pi/2$, π and $3\pi/2$.

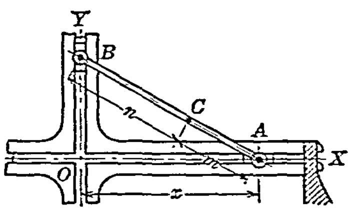
Ans. (1) $\omega = 6/5\pi$ rad./sec., clockwise; $v_m = 188$ in./sec.; (2) $\omega = 0$; $v_m = 377$ in./sec.; (3) $\omega = 6/5\pi$ rad./sec., counterclockwise; $v_m = 188.4$ in./sec.; (4) $\omega = 0$; $v_m = 377$ in./sec.

398. The gear K, 10 in. in diameter, and the crank OA, 10 in. long, can rotate about the shaft O. They are not connected together. The connecting rod AB, 50 in. long, has the gear L, 10



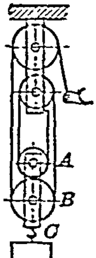
in. in diameter, rigidly attached to it. L is in mesh with K. K rotates at a uniform speed of 60 r.p.m., causing the crank OA to rotate. Find the angular velocity of the crank OA in the two vertical and two horizontal positions.

Ans. (1) $\omega = 10/11\pi$ rad./sec.; (2) $\omega = \pi$ rad./sec.; (3) $\omega = 10/9\pi$ rad./sec.; (4) $\omega = \pi$ rad./sec.



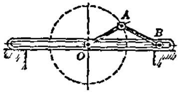
399. The sliding bar of an ellipsograph, $AB = l$, moves in slots along the axes of X and Y. The end A of the sliding bar undergoes harmonic oscillations $x = a \sin \omega t$, where $a < l$. $CA = m$ and $CB = n$. Find the velocity of C.

$$Ans.\quad v_C = \frac{a\omega}{l} \cos \omega t \sqrt{n^2 - m^2 + \frac{m^2 l^2}{l^2 - a^2 \sin^2 \omega t}}.$$



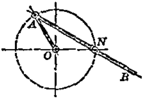
400. Find the space and body centrodes of the pulleys A and B when the weight C is being lifted. The radii of A and B are r_a and r_b, respectively.

Ans. The body centrodes are: a circle of radius r_a for A, and a circle of radius $\frac{1}{3}r_b$ for B. The space centrodes are vertical lines tangent to the body centrodes, on their right sides.



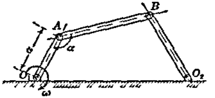
401. Find from geometrical considerations the space and body centrodes of the connecting rod AB The crank $OA = AB = r$

Ans The space centrode is a circle of radius $2r$, with its center in O, the body centrode is a circle of radius r with its center at the crankpin A

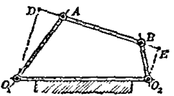


402 A rod AB is attached to a crank OA of radius r It passes through a pivoted guide N which is at a distance r from the crank axis O Find the centrodes of the rod

Ans A circle of radius r traced by A, and a circle of radius $2r$ with its center at the crankpin A

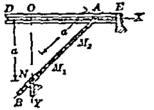


403 In the linkage shown in the sketch, O_1 and O_2 are fixed points The link O_1A of length a rotates about O_1 with an angular velocity ω Find by construction the point M on AB, where the velocity is directed along AB, and express this velocity as a function of the angle $O_1AB = \alpha$ *Ans* $v_M = a\omega \sin \alpha$



404 The linkage shown in the sketch has two fixed pins O_1 and O_2 The link O_1A rotates with an angular velocity ω_1 Find from the geometry of the system the angular velocity ω_2 of link O_2B Give it in terms of ω_1 and the distances O_1D and O_2E of the pins O_1 and O_2 from the center line of AB

Ans $\omega_2 = \omega_1 \dfrac{O_1D}{O_2E}$



405 A conchoidograph consists of a rod AB, one end of which moves in the slot DE The rod passes through a pivoted guide at N The distance between N and the center line of the slot DE is a Find the equations of the curves described by the points M_1

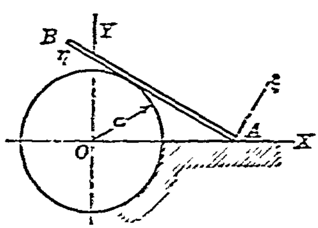
and M_2 on AB when $AM_1 = a$, and $AM_2 = a/2$ and the end A moves along the slot.

Ans. Path of M_1 is: $x_1^2y_1^2 = (a - y_1)^2(a^2 - y_1^2)$.
Path of M_2 is: $x_2^2y_2^2 = (a - y_2)^2(a^2 - 4y_2^2)$.

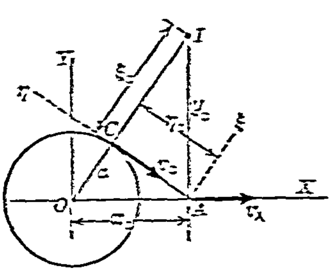
406. The point O_1 of a certain plane figure is moving to the right on a line parallel to OX with a velocity of 5 in. per sec. The distance between O_1 and OX is 15 in. The figure rotates about O_1 in a clockwise direction with a uniform angular velocity of $\frac{1}{3}$ rad. per sec. Find the space and body centrodes of the motion of the plane figure. Find the curve traced on the plane figure by a pencil fixed at $x = 0$, $y = 15$.

Ans. Axis OX and a circle of radius 15 in. with center at O_1. A spiral $\rho = (15\phi)$ in.

407. *The rod AB rests on a disc of radius a and its end A* moves on the line OX passing through the center of the disc. Find the equations of the centrodes of the rod.



Solution:



The instantaneous center I is the intersection of the perpendiculars AI and CI (§ 69). From geometrical considerations,

$$AI : OA = AC : OC,$$

or

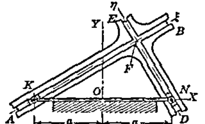
$$x_c^2(x_c^2 - c^2) = c^2y_c^2$$

is the equation of the space centrode. Also,

$$AC : CI = OC : CA, \quad \text{or} \quad \eta_c^2 = c\xi_c$$

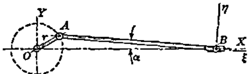
is the equation of the body centrode.

408. Assuming in the previous problem $a = 15$ in., $AB = 30$ in., find the velocity v of the point B when $OA = 25$ in. The velocity of A in this position is 10 in. per sec. and is directed positively along OX. *Ans.* $v_B = 8.5$ in./sec.



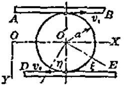
409. A plane figure has two slots cut in it perpendicular to each other. A pin K fits in slot AB and another pin N fits in the slot ED. $KN = 2a$. At time $t = 0$, AB coincides with the line KN. Find the equations of the centrodes for this motion.

Ans. $x_c^2 + y_c^2 = a^2$ and $\xi_c^2 + \eta_c^2 = 4a^2$.



410. The length of a connecting rod AB is so great compared with the radius r of the crank OA that the angle α is always small. Find the approximate equations of the body and space centrodes of the connecting rod AB under the assumption that $\sin \alpha = \alpha$ and $\cos \alpha = 1$ for all possible values of α.

Ans. $(x_c^2 + y_c^2)(x_c - l)^2 = r^2x_c^2$, and $(l\xi_c + \eta_c^2)(r^2\eta_c^2 - l^2\xi_c^2) = l^4\xi_c^2\eta_c^2$.



411. Two parallel racks AB and DE move in opposite directions with constant speeds v_1 and v_2. Both racks are in mesh with a gear of radius a. Find the equations of the centrodes of the gear-disc. Find the velocity v_0 of the gear-center O' and the angular velocity ω of the gear.

$$Ans. \quad y_c = a\frac{v_1 - v_2}{v_1 + v_2} \quad \text{and} \quad \xi_c^2 + \eta_c^2 = a^2\left(\frac{v_1 - v_2}{v_1 + v_2}\right)^2;$$

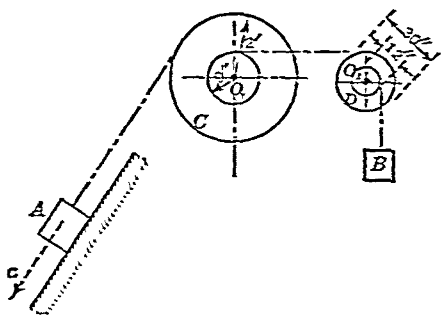
$$v_0 = \frac{v_1 - v_2}{2}; \; \omega = \frac{v_1 + v_2}{2a}.$$

412. A top spins on the platform of a car which is moving at a velocity of 25 ft. per sec. The axis of the top is vertical and it rotates at a speed of 30 revolutions per sec. Find the axodes of the absolute motion of the top.

Ans. A vertical plane parallel to the rails at a distance 1.59 in. from the top's axis, and a vertical cylinder of radius 1.59 in.

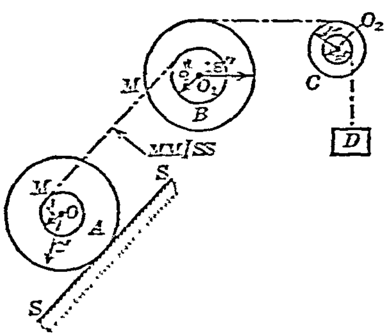
413. The counterweight A moves along the slide with an acceleration a in./sec.^{2}. Through a system of drums rotating on

fixed axes O_1 and O_2, it lifts the weight B. The cord holding A is parallel to the slide. Express the acceleration of the weight B



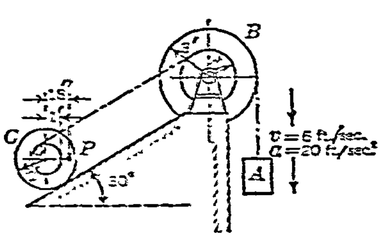
and the drums C and D in terms of the acceleration a.

Ans. $a_B = 0.15a$; $\alpha_C = 0.5a$ rad./sec.^{2}; $\alpha_D = 0.3a$ rad./sec.^{2}.



414. A body A rolls on the fixed plane. Two bodies B and C rotate about fixed axes. Determine the velocities of B and C in terms of the velocity of D. Also find the velocity of the center of A and the angular velocity of A in terms of the velocity of D.

Ans. $\omega_B = 4/3v$ rad./sec.;
$\omega_C = 2v$ rad./sec.;
$\omega_A = \frac{1}{3}v$ rad./sec.;
$v_0 = 2/3v$, where v is the velocity of D in ft./sec.



415. At a certain instant, A is moving downward with a velocity of 6 ft./sec. and it has a downward acceleration of 20 ft./sec.^{2}. Find the absolute velocity and absolute acceleration of point P, 18 inches horizontally to the right of O on the drum C. *Note:* The cord BC is parallel to the plane, at 30° to the horizontal, and the cylinder C rolls without slipping.

Ans. $v_p = 2.40$ ft./sec.; $a_p = 5.52$ ft./sec.^{2}.

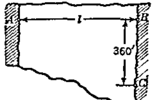
RELATIVE MOTION

15 Relative Motion

416 A river steamer plies between two cities 48 miles apart The trip up-stream takes 9 hrs , the down stream trip takes 4 hrs Find the velocity v of the river and the velocity u of the steamer in still water *Ans* $u = 8\frac{2}{3}$ mi /hr , $v = 3\frac{1}{3}$ mi /hr

417 A river $\frac{1}{2}$ mile wide flows between parallel banks with a velocity of 2.5 mi per hr A boat crossing the river in a direction perpendicular to the banks takes $4\sqrt{3}$ min to reach the other side Neglecting the variation of the river velocity near the banks, find the velocity u of the boat relative to the water

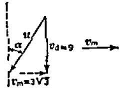
Ans $u = 5$ mi per hr



418 A river flows between parallel banks A boat steering straight across the river goes from A to C on the opposite bank in 10 min AB is perpendicular to the banks and C is 360 ft below B In order to land at B when starting from A, the boat must be steered up-stream at an angle to AB, the trip taking 12.5 min Find the width l of the river, the velocity u of the boat, and the velocity v of the river

Ans $v = 36$ ft /min , $u = 60$ ft /min , $l = 600$ ft

419 A rain-drop falling vertically has a velocity of 9 ft per sec near the earth Find its velocity relative to a man walking at a speed of $3\sqrt{3}$ ft per sec Find the angle α at which the rain hits the man



Solution

The velocity u of the drop relative to the man is the vector difference (§ 71) between its absolute velocity v_d and the absolute velocity v_m of the man

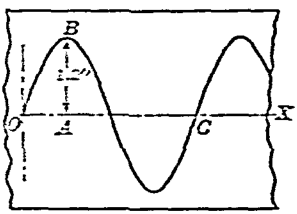
$$u = \sqrt{81 + 27} = 10.4 \text{ ft /sec}$$

Since $\tan\alpha = \frac{1}{3}\sqrt{3}$ $\alpha = 30°$

420 A rain-drop falling vertically traces a path on the side window of an automobile at an angle of 40° to the vertical The speed of the automobile is 36 miles per hour Find the absolute velocity v of the rain-drop *Ans* $v = 63$ ft per sec

421. A straight pipe moves parallel to itself in a direction perpendicular to its axis. Its speed is 10 in. per sec. Inside the pipe, a ball is moving along the axis in such a manner that its distance from a fixed point on the axis is $d = 2 \sin 2\pi t$. Write the equations of the path, the velocity, and the acceleration of the absolute motion of the ball.

Ans. The path is $y = 2 \sin \pi x/5$;
$v = 2\sqrt{25 + 4\pi^2 \cos^2 2\pi t}$ in. per sec.;
$a = -8\pi^2 \sin 2\pi t$ in. per sec.^{2}.



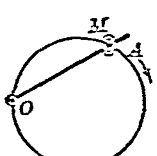
422. The chart of a vibration-recording instrument moves to the left with a velocity of $6\frac{1}{4}$ ft. per sec. The pen scribes a sinusoidal line on the chart with a maximum ordinate $AB = 1.2$ in. The distance $OC = 3$ in. Taking $t = 0$ at the point O, give the equation of the recorded motion.

Ans. $y = (1.2 \sin 50\,\pi t)$ inches.

423. At the Paris Exposition there was a circular platform revolving on concentric rails at a speed of 2 revolutions per hr. A man walking on the platform on a circular path 540 ft. from the center at a speed of 1.884 ft./sec. moved in a direction opposite to the motion of the platform. Find the absolute velocity v of the man. *Ans.* $v = 0$.

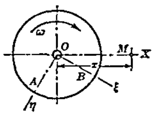
424. A train moves at a speed of 24 mi. per hr. A signal light hung 16.1 ft. above the ground, on a bracket attached to the last car, breaks loose and falls. Find the path of the absolute motion of the falling lamp, and the distance s traversed by the train before the lamp reaches the ground.

Ans. Parabola $y = 0.013x^2$; $s = 35.2$ ft.



425. A small ring M is put on a circular wire loop of 10 inches radius. A rod OA passes through the ring and rotates about the point O on the loop. Its angular velocity is constant and it rotates through a right angle every 5 seconds. Find the velocity v and the acceleration a of the ring.

Ans. $v = 2\pi$ in./sec.; $a = 0.4\pi^2$ in./sec.^{2}.



426. The motion of a point M along the line OX is defined by the equation $x = a \sin kt$, where x is the distance from O. A disc rotates about O, its center, with an angular velocity ω. Find the equation of the relative motion of M with respect to the disc.

Solution:

Assuming that at $t = 0$ axis $O\xi$ coincides with OX, angle ξOM is $\phi = \omega t$. $OM = x = a \sin kt$. The two equations define the relative motion in polar coordinates x, ϕ. Eliminating t, the path is $x = a \sin k(\phi/\omega) = a \sin (k/\omega)\phi$. When $\omega = k$, this becomes a circle of diameter a.

In orthogonal coordinates ξ, η, the path is (§ 56a)

$$\xi = (a \sin kt) \cos \omega t,$$
$$\eta = -(a \sin kt) \sin \omega t.$$

Transforming to eliminate t (§ 48), we find

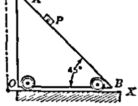
$$\cos \omega t = \frac{\xi}{a \sin kt}; \quad \sin \omega t = -\frac{\eta}{a \sin kt} = \xi^2 + \eta^2 = a^2 \sin^2 kt.$$

On the other hand,

$$\tan \omega t = -\frac{\eta}{\xi} = \omega t = -\tan^{-1}\left(\frac{\eta}{\xi}\right),$$

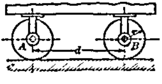
and

$$\xi^2 + \eta^2 = a^2 \sin^2 \left(\frac{k}{\omega} \tan^{-1} \frac{\eta}{\xi}\right).$$



427. A plane inclined at 45° to the horizontal moves to the right with a constant acceleration of 1 in./sec.². A body P slides down the plane with a constant relative acceleration of $\sqrt{2}$ in./sec.². The initial velocities of the plane and the body are both zero and the initial position of the body is $x = 0$, $y = h$. Find the path, velocity v, and the acceleration a of the absolute motion of P.

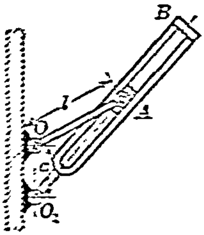
Ans. Straight line $y = h - \frac{1}{2}x$; $v = 2.24t$ in./sec.; $a = 2.24$ in./sec.².



428. Find the relative velocity u of the center of wheel A with respect to the other wheel B. The radii of both wheels equal r, the wheel base $AB = d$. The

car moves with a velocity v. Prove that the relative velocity of all points on A with respect to the wheel B is the same.

Ans. $u = v\dfrac{d}{r}$, normal to AB.



429. A crank-and-lever shaper mechanism consists of two parallel shafts O and O_1 and two cranks OA and O_1B. The end A of OA slides in a slot on O_1B. The distance $OO_1 = a$; the length of the crank $OA = l$; $l > a$. The shaft O rotates at a constant angular velocity and drives the shaft O_1. Find: (1) the angular velocity ω_1 of O_1 as a function of the distance $O_1A = s$; (2) the maximum and minimum values of ω_1; (3) the position of the shafts when $\omega = \omega_1$.

$$\textit{Ans.}\quad (1)\ \omega_1 = \frac{\omega}{2}\left(1 + \frac{l^2 - a^2}{s^2}\right);\quad (2)\ \max \omega_1 = \omega\frac{l}{l-a};$$

$$\min \omega_1 = \omega\frac{l}{l+a};\quad (3)\ \text{when } O_1B \perp OO_1.$$

430. A point moves with uniform velocity u along the circumference of a disc. The disc rotates around its axis in the opposite direction with an angular velocity ω. The radius of the disc is c. Find the absolute acceleration of the point.

Ans. $a = \dfrac{(u - c\omega)^2}{c}$, toward the center of the disc.

431. A disc of radius r ft. starts from rest and rotates around its axis with constant angular acceleration of n rad./min.². A point moves in the opposite direction along the circumference of the disc with a constant velocity u ft. per min. Find the absolute velocity and acceleration of the point.

$$\textit{Ans.}\quad v = rnt - u \text{ ft./min.};\ a = \sqrt{r^2n^2 + \frac{1}{r^2}(u - rnt)^4} \text{ ft./min.}^2.$$

432. A point moves with uniform velocity u along a chord of a disc which rotates in the same direction around its axis with a constant angular velocity ω. Find the velocity and acceleration of the absolute motion of the point at the moment when it is at the shortest distance h from the center of the disc.

Ans. $v = h\omega + u$; $a = h\omega^2 + 2u\omega$.

433. The motion of a point is defined by the equations $x = 3t$, $y = \frac{1}{2}gt^2$. A line l, passing through the origin of the coordinate system, rotates about that point in a counterclockwise direction at a speed of one revolution in 3 seconds. At time $t = 0$ the line coincides with the y axis. Find the projection v_l of the velocity of the point on the line l.

Ans. $v_l = gt\cos\frac{2\pi}{3}t - 3\sin\frac{2\pi}{3}t$

434. The minute hand of a chronometer is 1 in. long. Considering its motion from the instant that it is pointing to 12 o'clock, find the projection v_s of the velocity of the end of the minute hand on the direction of the second hand.

Solution

The velocity of the end of the minute hand is (§ 64a) $v = \frac{\pi}{1800}$ in./sec. $v_s = v\cos\alpha$ (§ 73), where α is the angle between v and the second hand.

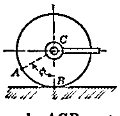
$$\frac{d\alpha}{dt} = \omega_{\text{second hand}} - \omega_{\text{minute hand}} = \frac{2\pi}{60} - \frac{2\pi}{3600} = \frac{59\pi}{1800}\ \text{rad./sec}$$

Integrating, with $\alpha_0 = -\frac{\pi}{2}$ at $t = 0$ we find $\alpha = -\frac{\pi}{2} + \frac{59\pi}{1800}t$. Therefore

$$v_s = \frac{\pi}{1800}\sin\frac{59\pi}{1800}t\ \text{in./sec}$$

435. A point moves on the circumference of a circle of radius r at a uniform velocity $r\omega$. A line l pivoted at the center of the circle rotates in a direction opposite to the rotation of the point. It revolves k times faster than the point. At the time $t=0$ the point is on the line. Find the projections v_l and a_l of the velocity and acceleration of the point on the line.

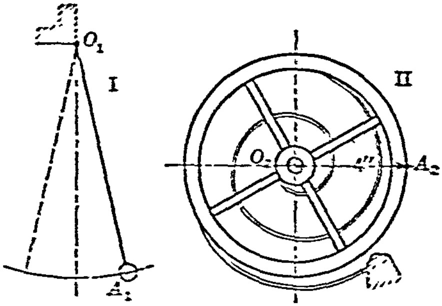
Ans. $v_l = r\omega\sin[(k+1)\omega t]$, $a_l = -r\omega^2\cos[(k+1)\omega t]$



angle $ACB = \phi$

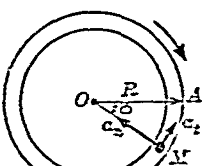
436. The wheel of a car moving at a speed of 36 mi./hr. rolls in a clockwise direction without slipping on the rail. v_r is the projection of the velocity of a point A on the rim of the wheel on the direction of the radius CA. Find the value of v_r as a function of the angle $ACB = \phi$. The wheel touches the rail at B.

Ans. $v_r = -52.8\sin\phi$ ft./sec



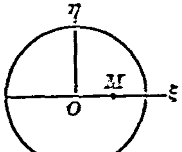
437. I is a pendulum consisting of a weight A_1 suspended on a thread O_1A_1. II is a small flywheel of radius $r = 4$ in. attached to a spiral spring. They are both oscillating harmonically about the centers O_1 and O_2, respectively. Their periods $T_1 = T_2 = \frac{1}{2}$ sec. The angular amplitude of the pendulum is $\pi/100$ radians and that of the flywheel is $\pi/2$ radians. The point A_2 on the rim of the flywheel swings over the lower half of the circumference and moves in phase with the weight A_1 on the pendulum. Find the projection v_1 of the velocity of the point A_2 on the line O_1A_1 as a function of time.

Ans. $v_1 = -(8\pi^2 \sin 4\pi t)\cdot\sin(0.49\pi \cos 4\pi t)$.



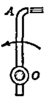
438. A round tube bent into a ring of radius $R = 1$ ft. rotates around the center O in the clockwise direction with a constant angular velocity $\omega = 1$ rad./sec. A small ball oscillates about the point A in the tube. The angle subtended by its path relative to the tube is $\varphi = \sin \pi t$. Find the normal and tangential components a_n and a_t of the ball's acceleration when $t = 2\frac{1}{6}$ sec.

Ans. $a_n = 13.8$ ft./sec.^{2}; $a_t = 4.9$ ft./sec.^{2}.



439. A disc of 1 in. radius starts from rest and rotates around its center in a clockwise direction with a constant angular acceleration of 1 rad./sec.^{2}. A point M oscillates on one of the diameters. Its distance from the center $OM = \xi$ is given by the equation $\xi = \sin \pi t$ in. Find the projections a_ξ and a_η of the absolute accelerations of the point M at $t = 1\frac{2}{3}$ sec.

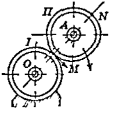
Ans. $a_\xi = \dfrac{\sqrt{3}}{2}\left(\pi^2 + \dfrac{25}{9}\right)$ in./sec.^{2}; $a_\eta = \left(\dfrac{\sqrt{3}}{2} - \dfrac{3}{5}\pi\right)$ in./sec.^{2}.



440. In a lawn sprinkler a stream of water flows through a pipe AO which is rotating about a vertical axis O with a speed of 60 r.p.m. Find the Coriolis acceleration a_{cor} at a point where the relative velocity (between the water and the pipe) is $u = 21/11$ ft./sec. in the direction OA.

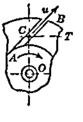
Solution

The Coriolis acceleration is (§ 72) $a_{cor} = 2u\omega$ where ω is the angular velocity of the sprinkler. $\omega = 2\pi$ rad./sec. $a_{cor} = 2 \times \frac{21}{11} \times 2\pi = 24$ *ft./sec.*^{2}, normal to OA, directed to the left.



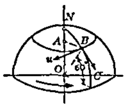
441. The crank OA rotates around the center O with a uniform angular velocity ω. Gear II of radius r can rotate around the pin at A and is in mesh with the fixed gear I of equal radius. Find the values and directions of the accelerations of the points M and N on gear II, which are the ends of the diameter parallel to the crank.

Ans. $a_M = 2r\omega^2$, in the direction MA, $a_N = 6r\omega^2$, in the direction NA.



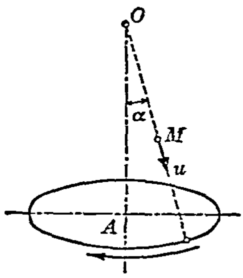
442. A turbine wheel with straight vanes rotates around its axis O with a constant angular velocity $\omega = 4\pi$ rad. per sec. The water flows between the blades with a uniform relative velocity $u = 6$ ft./sec. Find the radial and tangential components v_r and v_t of the absolute velocity, a_r and a_t of the absolute acceleration of the particle of water where $OC = 1.5$ ft. and the angle between the channel AB and OC is 45°.

Ans. $v_r = 4.2$ ft./sec., $v_t = 23.1$ ft./sec., $a_r = 342$ ft./sec.^{2}, $a_t = 107$ ft./sec.^{2}



443. The Yukon River flows with a velocity $u = 3$ mi. per hr. from East to West along the parallel of latitude 60°N. The radius of the earth is $R = 4000$ mi. Find the projection p of the acceleration of a water particle in the river on the direction of the tangent BC, considering only the acceleration due to the velocity u.

Ans. $p = 0.00055$ ft./sec.^{2}



444. A point M moves down an element of a right circular cone with a uniform velocity u. OA is the axis of the cone and $\angle MOA = \alpha$. At time $t = 0$ the distance $OM = c$. The cone rotates about its axis with a uniform angular velocity ω. Find the absolute acceleration of M.

Ans. $a = \omega \sin \alpha \sqrt{(c + ut)^2\omega^2 + 4u^2}$.

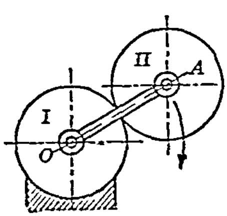
ROTATION OF RIGID BODIES

16. Composition of Rotations.

445. Two gears I and II of radii r_1 and r_2 are in mesh and rotate about fixed centers. Find the ratio between the angular velocities ω_1 and ω_2 of the two gears. Find the relative angular velocity $\omega_{1,2}$ between gear II and gear I for external and internal engagement.

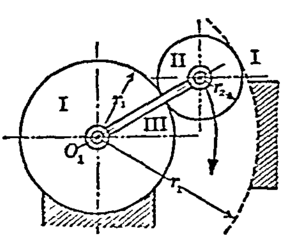
Ans. External engagement: $\dfrac{\omega_2}{\omega_1} = -\dfrac{r_1}{r_2}$; $\omega_{1,2} = \omega_1 \dfrac{r_1 + r_2}{r_2}$;

Internal engagement: $\dfrac{\omega_2}{\omega_1} = \dfrac{r_1}{r_2}$; $\omega_{1,2} = \omega_1 \dfrac{r_2 - r_1}{r_2}$.



446. The pin A of a crank OA carries a freely mounted gear II which is in mesh with the immovable gear I of the same radius having its center at O. How many revolutions will the gear II make around the pin A while the crank OA makes one turn around O?

Ans. One revolution.



447. A crank III connects the shafts of two gears of radii r_1 and r_2 which are in engagement. The engagement may be external or internal. Gear I is immovable. Crank III rotates about O_1 with an angular velocity of ω_3. Find the absolute angular velocity ω_2 of gear II and the relative angular velocity $\omega_{2,3}$ between gear II and the crank III.

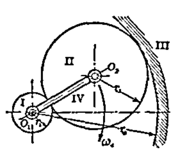
Ans. External engagement:

$$\omega_2 = \omega_3 \frac{r_1 + r_2}{r_2}; \quad \omega_{2,3} = \omega_3 \frac{r_1}{r_2};$$

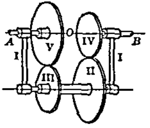
Internal engagement:

$$\omega_2 = -\omega_3 \frac{r_1 - r_2}{r_2}; \quad \omega_{2,3} = -\omega_3 \frac{r_1}{r_2}.$$

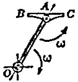
448. The gearing used to produce high speed rotation of a grindstone is made as follows. The crank *IV* is turned by means of a handle around O_1 with an angular velocity ω_4. A pin at the end of IV carries a wheel II of radius r_2 which is wedged between wheel I and the internal wheel *III*. The rotation of the crank causes II to roll on the inside of III and the rotation of II is transmitted by friction to the wheel I of radius r_1 which is attached rigidly to the spindle of the grindstone. Given r_3, find r_1 such that the speed ratio $\omega_1/\omega_4 = 12$ will exist. *Ans.* $r_1 = 1/11\, r_3$.



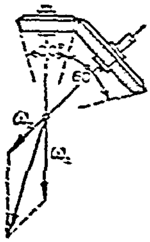
449. A frame I rotates with an angular velocity ω_1 around a fixed shaft AB. Two gears II and III are rigidly connected together and rotate about a shaft in the frame I parallel to AB. The gear II engages with an immovable gear IV and the gear III engages with the gear V, which can rotate around OA. The radii r_2, r_3, r_4, and r_5 are given. Find the angular velocity of the gear V. *Ans.* $\omega_5 = \omega_1 \left(1 - \dfrac{r_3 r_4}{r_2 r_5}\right)$.



450. The crank OA rotates with an angular velocity ω around a fixed axis O. The pedal BC rotates around A with the same angular velocity ω, but in the opposite direction. Find the absolute motion of the pedal.



Ans. Motion of translation; each point of the pedal describes a circle of radius OA.

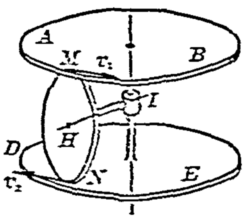


451. Two bevel gears with fixed axes have angles $\alpha = 30°$ and $\beta = 60°$. The first gear rotates with a speed of $\omega_1 = 10$ r.p.m. Find the angular velocity ω_2 of the second gear.

Ans. $\omega_2 = 0.173\pi$ rad./sec.

452. The bevel gear I has k_1 teeth. The bevel gear II has k_2 teeth. They are in mesh and their axes of rotation are mutually perpendicular. The gear I rotates at a speed of n_1 r.p.m. Find the relative angular velocity of the gears.

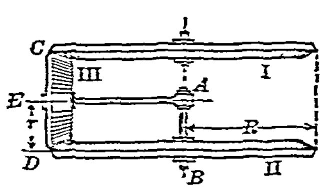
Ans. $\omega_{1,2} = \dfrac{\pi n_1}{30}\sqrt{1 + \left(\dfrac{k_1}{k_2}\right)^2}$ rad./sec.



453. A differential friction transmission consists of two discs AB and DE, free to rotate around the same shaft, and of a third disc MN wedged between them. MN rotates about an axis HI, which is perpendicular to the shaft. The radius of MN is $r = 2$ in. The velocity v_1 at M is 10 ft./sec. and the velocity v_2 at N is $12\frac{1}{2}$ ft./sec. Find the velocity v of the center H and the angular velocity ω of MN around HI.

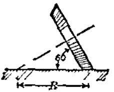
Ans. $v = 1.25$ ft./sec., to left; $\omega = 67.5$ rad./sec.

454. The shaft on which the bevel gear III rotates can swing around AB. The gear III is in mesh with the bevel gears I and II, which rotate around AB with angular velocities ω_1 and ω_2.



The radius of III is r. I and II have equal radii R. Find the angular velocity ω with which the axis of III swings around AB and the angular velocity ω_3 of III around its axis.

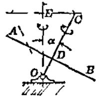
Ans. $\omega = \dfrac{\omega_1 + \omega_2}{2}$; $\omega_3 = \dfrac{\omega_1 - \omega_2}{2} \times \dfrac{R}{r}$.



455 A disc of radius r rolls around the circumference of a circle of radius R, 5 times a minute. Its plane is always at 60° to the plane of the circle. Find the angular velocity ω of the rotation of the disc around its axis and the angular velocity ω_1 of its rotation around the instantaneous axis.

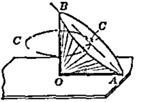
Ans $\omega = \frac{\pi}{3}$ rad /sec , $\omega_1 = \frac{\pi}{6}\sqrt{3}$ rad /sec

17 Rotation of a Rigid Body about a Fixed Point



456 A merry go round consists of a round platform AB 30 ft in diameter rotating around a central shaft OC. The platform rotates at a speed of 6 r p m. The shaft OC is inclined at an angle of $\alpha = 20°$ to the vertical and swings around the vertical center line in the same direction as the platform rotation at a speed of 10 r p m. $OD = 6$ ft. Find the velocity v of B when it is in its lowest position.

Ans $v = 26.3$ *ft /sec*



457 A right circular cone of altitude $CO = 18$ in with the angle at the vertex $AOB = 90°$, rolls on a plane. The vertex remains immovable at the point O. The center C of the base moves in a circle at the uniform speed of one revolution per second. Find the velocity of the ends A and B of the diameter AB.

Solution

The velocity v_A of A, as well as that of O, is zero. $v_A = 0$. OA is the instantaneous axis of the cone (§§ 79, 80). The distance BO is twice the distance from C to OA. Hence we have $v_B = 2v_C = 2 \times (OC \sin 45° \times 2\pi)$ $= 160$ in /sec

458. A body rotates around a fixed point. At a certain moment its angular velocity is given by a vector whose projections on the axes are $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{7}$ rad /sec. Find the velocity v at the same moment of a point whose coordinates are $\sqrt{12}$, $\sqrt{20}$, and $\sqrt{28}$ in.

Ans $v = 0$

459. The angular velocity of a body is $\omega = 7$ rad. per sec. Its instantaneous axis of rotation has the direction-angles α, β, and γ, ($<\pi/2$), where $\cos \alpha = 2/7$ and $\cos \gamma = 6/7$. Find the velocity v and its projections v_x, v_y, and v_z of a point whose coordinates are 0, 6, and 0 feet. Find the distance h of this point from the instantaneous axis.

Ans. $v_x = -36$ ft./sec.; $v_y = 0$; $v_z = 12$ ft./sec.; $v = 12\sqrt{10}$ ft./sec.; $h = 5.4$ ft.

460. The angular velocity of a body is $\omega = 6$ rad. per sec. Its instantaneous axis of rotation has the direction-angles α, β, and γ, where $\cos \alpha = \frac{1}{3}$, $\cos \beta = \frac{2}{3}$. Find a point in the plane $z = 0$, the projections of whose velocity on the x and y axes are $v_x = v_y = 6$ ft. per sec. *Ans.* $x = -1\frac{1}{2}$ ft.; $y = 1\frac{1}{2}$ ft.

461. A body rotates around a fixed point. The projections of the velocity of point M_1 (0, 0, 2) on the axes are $v_x = 1$, $v_y = 2$, and $v_z = 0$. The direction cosines of the velocity of point M_2 (0, 1, 2) are $\frac{2}{3}$, $-\frac{2}{3}$, and $\frac{1}{3}$. The coordinates are in feet; the velocities are in ft. per sec. Find the equations of the instantaneous axis of rotation and the angular velocity ω of the body.

Ans. $x + 2y = 0$ and $3x + z = 0$; $\omega = 3.2$ rad./sec.

462. The rotation of a rigid body is described by the derivatives of the so-called Euler's angles $d\theta/dt = 0$, $d\phi/dt = n$, and $d\psi/dt = \alpha n$. When $t = 0$, $\theta = 60°$, $\phi = 0$, and $\psi = 90°$, (1) find the projections ω_x, ω_y, ω_z of the angular velocity on the x, y, and z axes; (2) find a value for the coefficient α such that the space axode will be the plane XOY.

Ans. (1) $\omega_x = \dfrac{n\sqrt{3}}{2} \cos \alpha nt$; $\omega_y = -\dfrac{n\sqrt{3}}{2} \sin \alpha nt$; $\omega_z = (\alpha + \frac{1}{2})$; (2) $\alpha = -\frac{1}{2}$.

PART III. DYNAMICS

RECTILINEAR MOTION

18. Rectilinear Motion.

463. *A man weighing 160 lbs. stands on the floor of an elevator,* moving upward. What is the reaction of the floor on his feet (1) when the velocity of the elevator is constant, and (2) when its velocity is increasing at a rate of 5 ft./sec.^{2}?
Ans. (1) 160 lbs.; (2) 185 lbs.

464. An automobile weighing 2400 lbs. can accelerate from 10 to 30 mi./hr. in 4 seconds. Neglecting the rolling and wind resistances, what should be the "tractive effort" between the wheels and the ground? *Ans.* 550 lbs.

465. When released, a balloon weighing 4000 lbs. had a lifting force of 250 lbs. Under the action of a horizontal wind pressure the balloon travels in a direction which makes an angle of 30° with the vertical. Find the force of the wind on the balloon.
Ans. 86.7 lbs.

466. In the previous problem, determine the horizontal and vertical components of the acceleration of the balloon.
Ans. $a_h = 0.70$ ft./sec.^{2}, $a_v = 2.01$ ft./sec.^{2}.

467. A boat weighing 540 lbs. and moving with a speed of 4 mi./hr. enters a sea-weed area and stops in 10 seconds. Assuming the resistance of the weeds to the boat's motion to be uniform, what is this force? *Ans.* 9.9 lbs.

468. A magnetic particle weighing 3.5 grams is drawn through *a solenoid with an acceleration of 4 mtr./sec.^{2}. What is the force* on the particle, in lbs.? *Ans.* 0.00315 lb.

469. A spring is compressed by a force of 49,050 dynes. Express this force in lbs. *Ans.* 0.11 lb.

470. A body weighing 4 lbs. moves on a straight line with uniform acceleration. The distance traversed by the body is $s = 19.3t^2$ in. Find the force acting on the body. *Ans.* 0.40 lb.

471. A body slides down a plane which is inclined at an angle of $\alpha = 30°$ to the horizontal. The initial velocity is zero. The coefficient of friction is $f = 0.02$. Find the time T taken to travel a distance $l = 128.8$ ft. *Ans.* $T = 4.1$ sec.

472. A body lying on a floor receives an initial velocity of 6 ft./sec. It moves on a straight line and retards uniformly, traveling 12 ft. before stopping. Find the frictional force per lb. weight acting on the body. *Ans.* 0.047 lb. per lb. weight.

473. An elevator weighing 800 lbs. moves down a shaft with a uniform acceleration. In the first 10 seconds, it drops 100 ft. Find the tension T in the cable carrying the cage.

Ans. $T = 750$ lbs.

474. A body weighing 2 lbs. oscillates on a horizontal line about a fixed point on the line. The distance of the body from the point at any time is given by the equation $s = 10 \sin \pi/2t$ in. Find the relationship between the force P acting on the body and the distance s. What is the maximum value of P?

Ans. $P = -0.0128s$ lb.; $P_{max} = 0.128$ lb.

475. A stone is dropped into a well. The sound of the impact of the stone on the bottom of the well is heard 6.5 sec. after it is dropped. The velocity of sound is 1120 ft./sec. How deep is the well? *Ans.* 579 ft.

476. A train weighing 322,000 lbs. starts on a horizontal track and moves with a uniform acceleration. After 60 seconds, its speed is 36 mi./hr. The frictional resistance is equal to 0.005 of the train's weight. Find the drawbar pull of the locomotive.

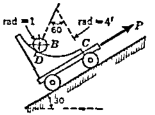
Ans. 10,430 lbs.

477. A body slides down a smooth plane which is inclined 30° to the horizontal. The body is started with an initial velocity of 6 ft./sec. How long will it take to slide 27 ft.? *Ans.* 1.50 sec.

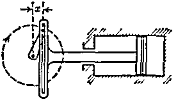
478. A body slides down a plane inclined 30° to the horizontal. It starts from rest and the frictional resistance is 0.1 of the body's weight. What is the velocity of the body after it has moved 6 ft.?

Ans. 12.43 ft./sec.

479. A train moves down an 0.8% grade at a speed of 24 mi./hr. The engineer applies the emergency brake suddenly. The total



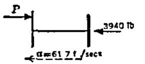
490. A car carrying a circular guide as shown weighs 480 lbs The smooth cylinder B weighs 96 lbs Under the action of a force P the car is drawn up the 30° plane so that arc CD subtends an angle of 60° Determine the reaction of the guide upon the cylinder, the acceleration of the system, and the force P. All rolling resistances and frictions are to be neglected *Ans* $P = 865$ lbs



491. This apparatus is being used to compress air. The crank is turning clockwise at 150 r p m The stroke is 18 inches The piston weighs 80 lbs and is 10 in in diameter The piston rod weighs 40 lbs Determine the force of the crank pin on the piston assembly when $x = 3$ in, the air pressure being 50 lbs/sq in at this instant.

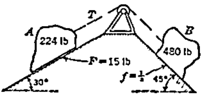
Solution

When the piston is in the position shown, it has an acceleration of $a = x\omega^2$ $= \frac{3}{12} \times (5\pi)^2 = 61\ 7$ ft/sec², toward the left The forces acting on the piston and rod which have components in the direction of motion are as shown in the free body diagram



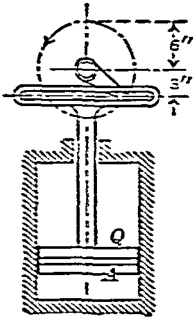
The equation of motion is (§ 92)

$$3940 - P = \frac{40 + 80}{32\ 2} \times 61\ 7, \quad P = 3710 \text{ lbs}$$



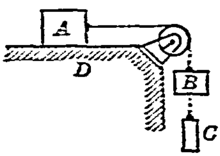
492 Two bodies, A and B, having weights as indicated, rest upon the inclined planes shown They are connected by a flexible inextensible cord T. Friction is as indicated Neglect the mass of the pulley At a given instant when the bodies are at rest, the system is allowed to move freely (*a*) Which way will body A move? (*b*) What is the acceleration of the system? (*c*) What is the tension in the cord T during the motion?

Ans (*b*) 4 54 ft/sec², (*c*) $T = 159$ lbs



493. The piston A weighs 300 lbs. The crank is rotating counterclockwise at 300 r.p.m. When the system is in the position shown, what is the force exerted on the piston rod at Q?

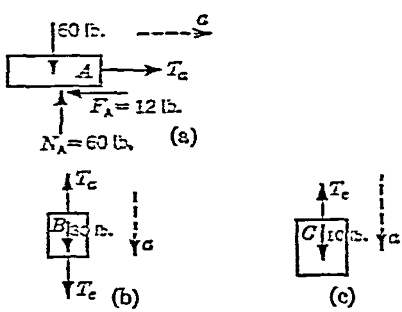
Ans. 2610 lbs.



494. A weighs 60 lbs., B weighs 30 lbs., and C weighs 10 lbs. The coefficient of friction between A and D is 1/5. Neglect the stiffness of the ropes, their masses, and that of the pulley. Find the tensions in the ropes and the acceleration of the system.

Solution:

Each of the three bodies, A, B, and C, is being accelerated by the forces shown acting in the free body diagrams (a), (b), and (c). Assuming the



acceleration of A to be a toward the right, the accelerations of B and C are both equal to a and directed as shown by dotted arrows in (b) and (c) (§ 92).

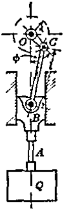
For body A: $\quad T_a - 12 = \dfrac{60}{32.2} \times a.$

For body B: $\quad T_c - T_a + 30 = \dfrac{30}{32.2}\,a.$

For body C: $\quad 10 - T_c = \dfrac{10}{32.2}\,a.$

Adding: $\quad 40 - 12 = \dfrac{100}{32.2}\,a, \quad a = 9.0 \text{ ft./sec.}^2$

Then $\quad T_a = 28.8$ *lbs.*, $\quad T_c = 7.2$ *lbs.*



495 A machine suggested by Reynolds for investigating the effects of rapidly changing compressive and tensile stresses upon materials consists of a reciprocating system The upper end of a test sample A is fixed in the cross head B A load Q of weight p is attached to its lower end The crank OC rotates about O with a constant angular velocity Neglecting the squares and higher powers of the ratio r/l of the crank length to the connecting rod length, find the longitudinal force T acting on A

Ans $T = p + (p/g)r\omega^2[\cos\omega t + (r/l)\cos 2\omega t]$

496 A street car oscillates harmonically in a vertical direction on its springs The amplitude of motion is 1 inch its frequency is 2 cycles per second The loaded cab weighs 20 000 lbs The truck and wheels weigh 2000 lbs Find the force acting on the rail

Ans Varies between 30,170 and 13 830 lbs

497 A sphere which weighs 1 gram falls under the action of gravity The air resistance is such that the equation of motion of the sphere is $x = 490t - 245(1 - e^{-2t})$ cm, where x is the distance from the starting point, at any time t Determine the air resistance as a function of the velocity v of the sphere

Ans $R = 2v$ (R in grams, v in cm /sec)

498 A body is dropped from a height h and falls to the ground Assuming the force of gravitation to be inversely proportional to the square of the distance from the center of the earth, find the time T in seconds taken to reach the surface of the earth and the velocity v at the instant it strikes the surface Neglect the effects of air resistance

$$Ans \quad T = \sqrt{\frac{R+h}{2gR^2}}\left(\sqrt{Rh} + \frac{R+h}{2}\cos^{-1}\frac{R-h}{R+h}\right),$$

$$v = \sqrt{\frac{2gRh}{R+h}}$$

499 A body weighing 45 lbs is thrown vertically upward with a velocity of 60 ft /sec The air resistance is 0 03v lbs , where v is the velocity of the body in ft /sec How soon will the body reach its highest position?

normal N to the sail plane, s is the sail area, which is 50 sq. ft., and $f = 0.0018\sqrt{2}$ is a constant obtained by experiment. The force P is normal to the sail ab. Neglecting friction, find the largest possible velocity of the ice-boat and the angle α which a pennant hung from the mast would make with the sail plane at this velocity. If the ice-boat started with zero velocity, how far would it have to travel before it reached a velocity $v = \frac{2}{3}w$?

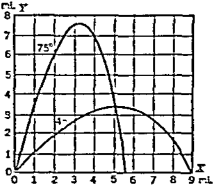
Ans. (1) $v_{max} = w$; $\alpha = 0$. (2) $s = 300$ ft.

CURVILINEAR MOTION

19. Curvilinear Motion.

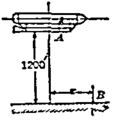
504. The motion of a body weighing $\frac{1}{4}$ lb. is given by the equations $x = 2t$ in., $y = 3 + t - 5t^2$ in.; find the force in lbs. acting on the body. *Ans.* $F_x = 0$; $F_y = -0.00647$ lb.

505. The motion of a particle weighing 2 oz. is given by the equations $x = 3\cos 2\pi t$, $y = 4\sin 2\pi t$, in inches. Find the projections of the force acting on the particle as functions of its coordinates. *Ans.* $F_x = -\dfrac{6\pi^2 x}{386}$ oz.; $F_y = -\dfrac{8\pi^2 y}{386}$ oz.



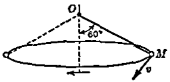
506. A 4-in. marine gun fires its shell weighing 38 lbs. with a muzzle velocity $v_0 = 2300$ ft. per sec. Actual trajectories of the shell are shown in the sketch: (1) for a gun elevation of 45° and (2) for an elevation of 75°. For both cases find the increase in altitude reached and distance traveled if air resistance were not acting.

Ans. (1) $\delta y = 7.8$ mi.; $\delta x = 31.2$ mi.;
(2) $\delta y = 14.5$ mi.; $\delta x = 15.6$ mi.



507. A dirigible A flies at an altitude of 1200 ft. with a velocity of 72 miles per hr. At what distance x before passing over a point B should a bomb be dropped from the dirigible to hit B? Neglect the effects of air resistance.

Ans. $x = 910$ ft.



511. A body M weighing 2 lbs. is suspended on a string 12 in. long. The other end of the string is fixed at O. The body M moves around a circular path on a horizontal plane, the string forming an angle of 60° with the vertical. Find the velocity v of the body and tension T in the string.

Ans. $v = 6.9$ ft./sec.; $T = 4$ lbs.

512. A stone weighing 6 lbs. tied to the free end of a string 3 ft. long moves around a circle in the vertical plane. The ultimate strength of the string in tension is 10 lbs. Find the angular velocity ω at which the string will break.

Ans. $\omega = 2.67$ rad./sec.

513. The rails of a railroad track are banked in the curves—that is, the outer rail is at a higher level than the inner rail. This is done so that the weight of a car and its centrifugal force in rounding the curve will have a resultant in the direction perpendicular to the plane of the track. Find the difference in level h between the outer and inner rails in a curve with a radius of 1200 ft. around which the trains are to run at a speed of 30 ft./sec. The rail gauge is 4 ft. 8½ inches. *Ans.* $h = 1.32$ in.

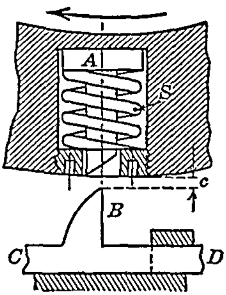
514. A load is weighed in the cab of a locomotive while it is rounding a curve at a speed of 48 mi. per hr. The load weighs 10 lbs. but the spring scales suspended from the roof of the cab show a reading of 10.2. Neglecting the effects of spring scale parts, find the radius of the curve. *Ans.* 770 ft.

515. A 4-lb. weight is suspended on a rope 3 ft. long. It is struck a blow which gives it a horizontal velocity of 15 ft. per sec. Find the tension in the rope just after the impact.

Ans. 13.3 lbs.

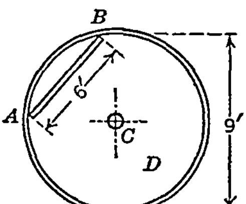
516. Find the maximum load on the pivot in the machine of Problem 624, assuming the weight of the hammer to be 40 lbs., its velocity at B being 20.4 ft./sec. *Ans.* 197 lbs.

517. What is the angle between the rod and the vertical in the impact testing machine of Problem 624 when the load on the pivot is equal to zero? *Ans.* $\phi = 48° 10'$.



518. A safety device to prevent the overspeeding of a flywheel acts as follows: A plunger A weighing 3 lbs. is held in the rim of the flywheel by a spring S. At the limiting speed, 120 r.p.m., the plunger A protrudes far enough to hit the lug B on the slider CD of an automatic stop. The clearance at rest is $c = 1$ in. The center of gravity of A at rest is 4.83 ft. from the axis of the engine shaft. Find the characteristic of the spring S if the initial compression of the spring is negligible. *Ans.* 72.5 lbs./in.

519. A man on a bicycle goes around a curve of 60 ft. radius with a velocity of 15 ft./sec. Find the angle between the plane of the bicycle and the vertical. *Ans.* $6° 39'$.



520. A rod AB weighing 10 lbs. rests on the horizontal table D. Its ends bear against a smooth circular rim on the edge of the table. The system is rotating about the table center C at a speed of 200 r.p.m. Compute the reactions at the ends of the rod. *Ans.* 307 lbs.

521. A particle of weight w moves on a catenary

$$y = \tfrac{1}{2}(e^x + e^{-x})$$

under the action of a force repelling it from the x axis. The force is proportional to $(w/g)y$. At $t = 0$, $x_0 = 1$, and $(v_x)_0 = 1$. Find the motion of the particle and the force it exerts on the restraining curve. *Ans.* $x = t + 1$; $F = 0$.

522. The radius of the earth is $R = 636 \times 10^6$ cm.; its average specific gravity is 5.5. The radius of the terrestrial orbit is approximately 148×10^{11} cm., and the period of rotation around the sun is 365.25 days. Find the mass M of the sun. *Ans.* 197×10^{31} grams.

523. A particle of weight w travels under the action of a central force F on a path whose equation is $r^2 = a \cos 2\phi$, which is a lemniscate, where r is the distance of the point from the center

of attraction and a is a constant. At the initial moment, $r = r_0$ and the velocity is v_0 directed at an angle α to the radius vector between the point and the center of attraction. Find the force F as a function of r.

Ans. $F = \dfrac{3wa^2}{gr^7} r_0^2 v_0^2 \sin^2 \alpha.$

524. A particle of weight w moves around a fixed point O under the action of a central force F which depends only on the distance $OM = r$. The velocity of the point is $v = a/r$, where a is a constant. Find the force F and the path of the particle's motion.

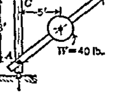
Ans. $F = -\dfrac{wa^2}{gr^3}$; the path is a logarithmic spiral.

525. A mass of 1 gram is attracted to a fixed point by a force which is inversely proportional to the cube of the distance between the mass and the point. At a distance of 1 cm., the force acting is 1 dyne. At time $t = 0$, $r_0 = 2$ cm., and the velocity $v_0 = 0.5$ cm./sec. is directed at an angle of 45° to the radius vector between the mass and the center of attraction. Find the motion of the mass.

Ans. $r = 2e^{\phi}$.

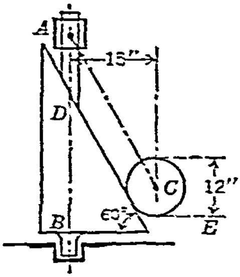
526. A particle M of 1 gram mass is attracted to a fixed center O by a force which is inversely proportional to the fifth power of the distance to the center. At a distance of 1 cm., the force is 8 dynes. At time $t = 0$ the particle is at a distance $OM_0 = 2$ cm. from the center of attraction, and its velocity $v_0 = 0.5$ cm./sec. is directed normally to OM_0. Find the path of the particle.

Ans. $r = 2 \cos \phi$. A circle of radius 1 cm.



527. The boom AB of a crane carries a load of 40 lbs. It is supported by a pin at A and held inclined by a horizontal cord BC, and revolves around a vertical axis AC with a constant speed of 12 rad./sec. Neglect the weight of the boom. (*a*) What is the tension in the cord BC? (*b*) What are the components of the reaction at A?

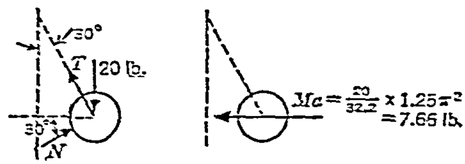
Ans. (*a*) 472.5 lbs.; (*b*) $R_h = 422.5$ lbs.; $R_v = 40$ lbs



528. A cylinder C weighing 20 lbs. rests upon the smooth inclined plane DE and is suspended by a cord AC which makes an angle of 30° with the vertical. The plane and cylinder are rotated about the vertical axis AB at a speed of 30 r.p.m. Determine the tension in the cord and the reaction of the plane on C.

Solution:

The cylinder C has an acceleration $r\omega^2$ directed toward the axis of rotation. The forces acting on the body C are shown in (a) and the effective force



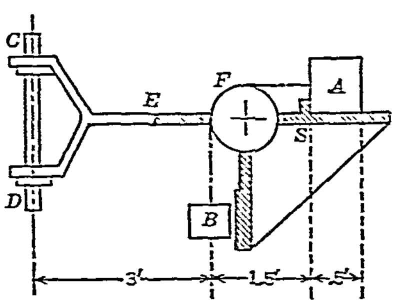
(a) (b)

system for body C is shown in (b). These two force systems are equivalent. Therefore

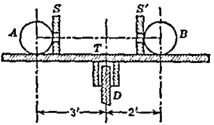
$$T \cos 30° - 20 + N \sin 30° = 0,$$
$$T \sin 30° - N \cos 30° = 7.66.$$

Solving, we find

$$T = 21.2 \text{ lbs.}, \qquad N = 3.4 \text{ lbs.}$$

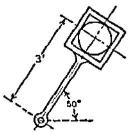


529. The frame E rotates about the vertical axis CD at a speed of 30 r.p.m. A is a body which weighs 10 lbs. and rests on E bearing against the stop S. B weighs 20 lbs. and is suspended by means of a cord that passes over pulley F and is fastened to A. Compute the force on the stop S under these conditions, neglecting friction. At what speed of rotation would B be lifted? If the coefficients of friction for the contact surfaces of A and B are each 0.25, what speed of rotation would be required to lift B? *Ans.* 45.1 r.p.m.

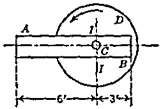


530. The ball A weighs 10 lbs. and B weighs 40 lbs. These balls lie upon a horizontal rotating table T, and bear against the stops S and S'. An elastic cord, the tension in which is 30 lbs. when the table is at rest, connects the two balls. What are the pressures against the stops when the table is rotated at 20 r.p.m. about the vertical peg at D?

Ans. $F_a = 25.9$ lbs.; $F_b = 19.1$ lbs.

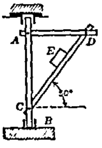


531. A smooth sphere, weighing 2 lbs., is in a box rigidly fastened to an arm which can be rotated about a horizontal axis (perpendicular to the plane of the figure). The system is caused to rotate counterclockwise so that its speed is uniformly increasing at the rate of 2 rev./sec./sec. When the arm is in the position shown, the speed of rotation is 3 rev./sec. What are the forces acting on the sphere at this instant? Represent them on a free body sketch of the sphere.



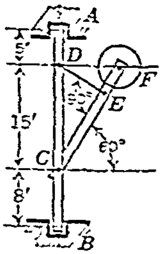
532. A board AB which weighs 20 lbs. rests upon a horizontal table D, rotating counterclockwise with it about the peg C at a constant speed of 400 r.p.m. Determine the internal force acting on the section I–I of the board.

Ans. 2180 lbs.

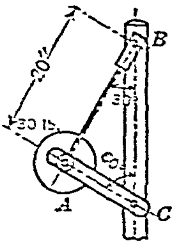


533. The frame shown in the sketch rotates at a constant speed of 200 r.p.m. about its vertical axis AB. The block E, weighing 100 lbs., rests upon the rough board CD in such a position that its center of gravity is 3 ft. from the upright AB. Determine the frictional and normal reactions on E, assuming that E does not slip on CD.

Ans. $R_n = 3200$ lbs.; $R_f = 2550$ lbs.

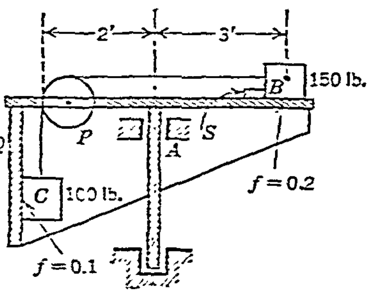


534. A flyball F weighs 50 lbs. and is carried by the bar CE, which is supported by a pin at C and a cord DE. When the whole system is rotated about the vertical bar AB at a constant speed of 10 rad./sec., determine all the forces acting on the bars AB and CE.

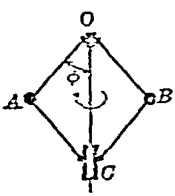


535. A conical pendulum rotates about the vertical spindle BC. The ball A, whose weight is 30 lbs., is held in position by the cord AB and by the link AC, which is pinned to the ball at A and pinned to the spindle at C. The angular velocity is $\omega = 4\pi$ rad./sec. Find the tensions in the cord AB and the link AC.

Ans. $T_b = 87.4$ lbs.; $T_c = 91.3$ lbs.



536. A frame is rotating about the vertical shaft A. A body B rests on a horizontal platform and bears against the stop S. B and C are connected by a cord which passes over pulley P. C hangs suspended under the platform and bears against the stop Q as the frame rotates. At what speed of rotation will body C start to rise? *Ans.* 3.1 rad. per sec.



537. A governor of Watt's type rotates around its vertical spindle with a constant angular velocity ω. All the links in the governor have the same length l. Find the angle between OA and the vertical. Consider only the effects of the weights p of each ball and the weight p_1 of the bushing C.

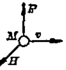
$$Ans.\quad \phi = \cos^{-1}\frac{(p + p_1)}{pl\omega^2}\,g.$$

538. A particle of mass m having a negative electric charge q enters a uniform electrostatic field of intensity E with a velocity v_0 normal to the direction of the field. Find the path of the particle in the field, where it is under the action of a force $F = qE$

opposite to the direction of the field. Neglect the action of gravity.

Ans. $y = \dfrac{1}{2}\dfrac{qE}{mv_0^2}x^2.$

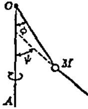
539. A particle of mass m, carrying a negative electric charge q, enters into a magnetic field of intensity H with a velocity v_0 normal to the direction of the field. Find the path of motion of the particle after it enters the field if the force acting upon it is $F = qHv$. F is directed perpendicularly to H and v, as shown in the sketch. Neglect the action of gravity.



Ans. A circle of radius $\dfrac{v_0 m}{qH}$.

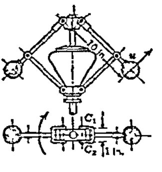
540. A particle of weight w oz., moving in a vertical plane, is attracted to a fixed point by a force which is proportional to the distance from the point. It is also acted upon by the force of gravity. The attraction to the point is k oz. at 1 in. distance. At time $t = 0$, $x = a$, $v_x = 0$, and $v_y = 0$. Give the equation of motion.

Ans. $y = \dfrac{w}{k} - \dfrac{w}{ka}x.$



541. A body M weighing $p = 200$ gr. is suspended on a string OM of length $a = 2$ in. The other end of the string is fixed at O on the vertical axis OA. Attached to M the string MM_1 of length $b = 2/\sqrt{3}$ in. carries on its free end the body M_1 weighing $p = 200$ gr. The system rotates around OA with a uniform angular velocity ω. The angles ϕ and ψ between the strings and the vertical are such that $\tan\phi = (4/5)\tan\psi$. Find the angles ϕ and ψ, the angular velocity ω, and the tensions T_a and T_b in the strings.

Ans. $\phi = 33°\,30'$; $\psi = 39°\,40'$; $T_a = 480$ gr.; $T_b = 260$ gr.



542. A governor is running steadily at a speed of 180 r.p.m. Due to a change in load, the engine speeds up and the balls move outward with a relative velocity of $u = 0.6$ ft./sec. The balls each weigh 20 lbs. Neglecting the weight of the linkage, find the additional forces on the bearings C_1 and C_2 due to the Coriolis acceleration.

NOTE: In the solution take the angles between the arms and the spindle to be 45° and consider the deviation from the normal speed as negligibly small.

Ans. $F = 99$ lbs.

543. A ring slides on a smooth rod 40 in. long. The rod rotates around one end in a horizontal plane. It has a speed of 60 r.p.m. At time $t = 0$, the ring is 30 in. from the center of rotation and its relative velocity is zero. Find the time t at which the ring will leave the rod.

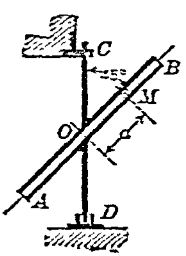
Solution:

No force component acts on the ring along the rod; the component a_r of the absolute acceleration a of the ring along the rod is zero. $a = a_{tr} \overrightarrow{+} a_{rel} \overrightarrow{+} a_{cor}$ (§ 72). a_{cor} acts normally to the rod, has no component along the rod; $a_{tr} = r\omega^2$ and is directed toward the center of rotation; a_{rel} is directed along the rod from the center. Therefore $a_r = a_{rel} - r\omega^2 = 0$; $a_{rel} = r\omega^2 = 4\pi^2 r$ in./sec.^{2}, where r is the distance of the ring from the center of rotation.

Since $a_{rel} = du/dt = d^2r/dt^2$, where $u = dr/dt$, is the relative velocity of the ring, $d^2r/dt^2 = 4\pi^2 r$, with $u = 0$, $r = 30$ at $t = 0$. Integrating (§ 85b), $d^2r/dt^2 = du/dt = u\,du/dr = 4\pi^2 r$, or $d(u^2) = 4\pi^2 d(r^2)$; $u^2 = 4\pi^2(r^2 - 30^2)$. From this equation $u = dr/dt = 2\pi\sqrt{r^2 - 900}$. Integrating once more, we have

$$t = \frac{1}{2\pi}\log\left(r + \sqrt{r^2 - 900}\right);$$

at $r = 40$ in., $t_1 = 0.18$ sec.



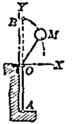
544. The pipe AB rotates around a vertical axis CD with a constant angular velocity ω. The angle between AB and CD is 45°. A heavy small ball M is in the pipe. At time $t = 0$, its velocity is zero and its distance from O is a. Neglecting the effect of friction, find the motion of the ball.

$$\textit{Ans.}\quad x = \frac{g\sqrt{2}}{\omega^2} + \left(a - \frac{g\sqrt{2}}{\omega^2}\right)\cosh\left(0.5\omega t\sqrt{2}\right).$$

545. A projectile weighing p lbs. is shot into the air at an angle α to the horizontal, with a velocity v_0. Assume the resistance R offered by the air to be proportional to the velocity: $R = fv$ lbs. Find the motion of the body under the action of gravitational and frictional forces.

$$\textit{Ans.}\quad x = \frac{pv_0\cos\alpha}{fg}\left(1 - e^{\frac{-fgt}{p}}\right),$$

$$y = \left(\frac{p}{fg}v_0\sin\alpha + \frac{p^2}{f^2g}\right)\left(1 - e^{\frac{-fgt}{p}}\right) - \frac{p}{f}t.$$



546. An elastic thread fixed at A passes through a smooth ring at O and has a ball M weighing m oz. attached to its free end. The free length of the thread is $l = AO$ and its spring characteristic is k oz. for 1 inch elongation. The thread is stretched along AB until its length is doubled and the ball M is given a velocity v_0 normal to AB. Neglecting the effect of gravity, find the path of the ball.

Ans. $\dfrac{x^2gk}{mv_0^2} + \dfrac{y^2}{l^2} = 1.$

547. A particle M of weight p is attracted to n fixed coplanar points $C_1, C_2, C_3, \cdots, C_n$ by *forces proportional to distances from* the points. The force of attraction to any center C_i $(i = 1, 2, 3, \cdots, n)$ may be written $k_i \times \overline{MC}_i$. Neglecting gravitational forces, find the motion of the point. At $t = 0$, $x = x_0$, $y = y_0$, $v_x = 0$, and $v_y = v_0$.

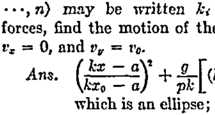
Ans. $\left(\dfrac{kx - a}{kx_0 - a}\right)^2 + \dfrac{g}{pk}\left[(ky - b) - \dfrac{ky_0 - b}{kx_0 - a}(kx - a)\right]^2 = 1$;
which is an ellipse; $a = \Sigma k_i x_i$; $b = \Sigma k_i y_i$; $k = \Sigma k_i$.

548. A particle M is attracted to two points C_1 and C_2 by forces proportional to the distances from the points: $f_1 = \lambda(MC_1)$ and $f_2 = \lambda(MC_2)$. The point C_1 is fixed at the origin of the coordinate system. The point C_2 moves with a constant velocity along the x axis: $x_2 = 2(a + bt)$. At time $t = 0$, M is in the xy plane with $x = y = a$ and the components of its velocity are $v_x = v_z = b$, $v_y = 0$. Find the path of the particle M.

Ans. $\dfrac{y^2}{a^2} + \dfrac{z^2}{b^2} \times \dfrac{2\lambda g}{p} = 1.$

MOTION OF RIGID BODIES

20. Principle of D'Alembert.



549. A system consists of three bodies connected by the inextensible cords shown. The weights and coefficients of friction are as shown. Neglect the masses of the pulleys. At a certain instant when the bodies are at rest, the system is allowed to move freely. (a)

What is the acceleration of the bodies? (*b*) What are the tensions in the cords T_1 and T_2?

Ans. $a = 1.55$ ft./sec.^{2}; $T_1 = 196$ lbs., $T_2 = 95.4$ lbs.

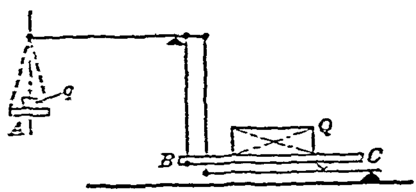
550. The roadbed of a railroad track compresses 1 in. under a load of 130 lbs./sq. in. A locomotive passes over the rails. Its driver rotates at a speed of 420 r.p.m. The static load of the driver is 14,000 lbs. and the driver has an unbalanced counterweight of 193.2 lbs. at a distance of 12 in. from the center of rotation. The load is transmitted to the roadbed by half of a tie which is 10 in. wide and 8 ft. 4 in. long. Neglecting the elastic effects of the rail and ties, find the range of deflections of the roadbed under the action of the moving load.

Ans. The deflection varies from 0.018 in. to 0.197 in.

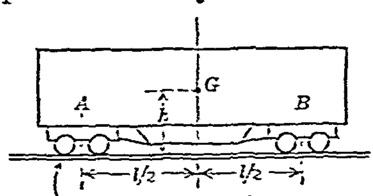
551. A locomotive moves with a uniform acceleration and attains a speed of 48 mi. per hr. in 20 seconds after it starts. Find the position of the water surface in the tender-tank.

Ans. A plane inclined to the horizontal at an angle $\alpha = 6° 15'$.

552. At the beginning of an upward grade, a highway makes a circular curve of 1800 ft. radius, in a vertical plane. A car weighing 3400 lbs. travels at a speed of 60 miles per hour. At rest, the springs of the car are compressed 6.8 inches under its weight. What is the deflection of the springs while the car passes the curve? *Ans.* 7.5 inches.



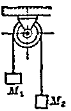
553. The platform *BC* of the scale shown schematically in the sketch weighs 120 lbs., while the weight tray *A* weighs 2 lbs. The box *Q* weighs 400 lbs. and is balanced by the weight *q*, which weighs 10 lbs. Neglecting the weight of the several rods, what is the acceleration of *A* if a ½ lb. weight is put on the tray? *Ans.* 3.4 ft./sec.^{2}.



554. A street car has a power truck *A* and a pony truck *B*. The center of gravity is at equal distances $l/2$ from the truck pivots, and is at an elevation h above the rails. The coefficient of friction

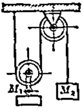
between the drivers of A and the rails is f. What is the maximum acceleration the motors can give to the car without having the wheels spin?

Ans. $\dfrac{fg}{2(1 - fh/l)}$.

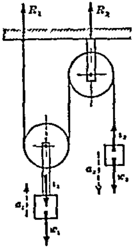


555. A flexible inelastic cord passes over a pulley and carries two loads M_1 and M_2 at its ends. M_1 weighs p_1 lbs. and M_2 weighs p_2 lbs $p_2 > p_1$. Find the acceleration a of the loads and the tension T in the cord. Neglect the inertia effects of the pulley.

Ans. $a = \dfrac{p_2 - p_1}{p_2 + p_1} g$, $T = \dfrac{2p_1p_2}{p_1 + p_2}$.



556. The system of pulleys shown in the sketch carries two loads M_1 weighing 10 lbs and M_2 weighing 8 lbs. Find the acceleration a_2 of the load M_2, neglecting the masses of the pulleys.



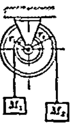
Solution

The acceleration a_1 of M_1 is equal to half the acceleration a_2 of M_2, $a_1 = a_2/2$. The sketch shows the external forces w_1 and w_2 and the inertia forces i_1 and i_2 acting on the system. All these forces are in static equilibrium (§ 91a) $w_1 + i_1 = 2(w_2 - i_2)$, or

$$w_1 + \frac{w_1}{g} a_1 = 2\left(w_2 - \frac{w_2}{g} a_2\right).$$

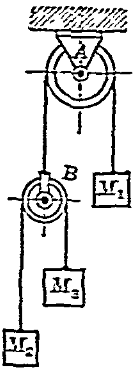
With $a_2 = 2a_1$,

$$a_2 = \frac{2w_2 - w_1}{2w_2 + \frac{w_1}{2}} g = 9.1 \text{ ft./sec}^2.$$



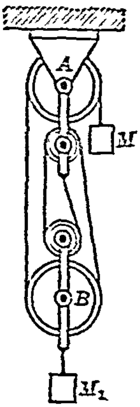
557. Two loads, M_1 weighing p_1 and M_2 weighing p_2, hang on two inelastic ropes wound on two pulleys which are rigidly mounted on a common axle. The radii of the pulleys are r_1 and r_2. The loads move under the action of gravity. Find the angular acceleration α of the pulleys, neglecting the effects of their own masses.

Ans. $\alpha = g \dfrac{p_2r_2 - p_1r_1}{p_2r_2^2 + p_1r_1^2}$.



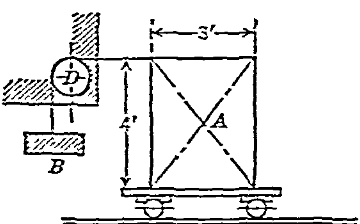
558. A system consisting of a fixed pulley A and a moving pulley B is shown in the sketch. The pulleys carry three loads M_1, M_2, and M_3 by means of inelastic cords, as shown. The loads weigh w_1, w_2, and w_3; $w_1 < (w_2 + w_3)$ and $w_2 \gtreqless w_3$. At a certain instant all the masses start to move. Neglecting the effects of the masses of the pulleys and strings, find the relationship between w_1, w_2, and w_3 for which the weight M_1 will move downward from rest.

$$Ans. \quad w_1 > \frac{4w_2w_3}{w_2 + w_3}.$$



559. An 80-lb. weight M lifts a load M_1 by means of a block and tackle. M_1 and the part of the tackle which moves weigh 300 lbs. The radii of the larger pulleys are r and those of the inner pulleys are r_1. The large pulleys each weigh 5 lbs. and the small pulleys each weigh 1 lb. Assuming the masses of the pulleys to be distributed around their rims, find the acceleration of the weight M. *Ans.* $a = 0.047$ g.

21. Translation of a Rigid Body.



560. A box A, 3 ft. × 2 ft. × 4 ft., weighs 1000 lbs. Assuming that the box will not slip on the carriage, what is the maximum weight that B may have without causing A to tip over when the acceleration of the carriage is 8 ft./sec.^{2} to the right? The friction and weight of pulley D may be neglected.

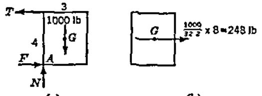
Solution:

The box shown is undergoing a motion of translation. At the instant that a tipping motion is impending, the forces acting on the body are as shown on page 200 in (a) and the effective force system is as shown in (b). The two force systems are equivalent (§ 92); therefore:

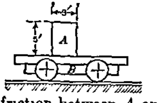
$$\Sigma M_A = 1000 \times 1.5 - 4T = 248 \times 2, \qquad T = 251 \text{ lbs.}$$

At this same instant body B has an acceleration of 8 ft /sec² upward

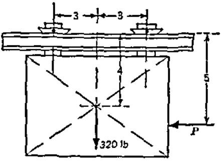
$$T - W_B = \frac{W_B}{32\ 2} \times 8, \qquad 251 - W_B = \frac{W_B}{32\ 2} \times 8 \qquad W_B = 202 \text{ lbs}$$



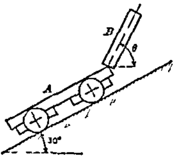
(a) (b)



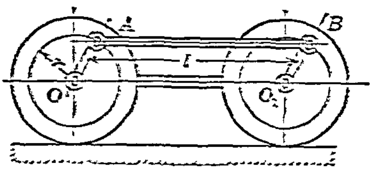
561. A is a rectangular prism weighing 2000 lbs which rests upon freight car B Assuming that A does not slip on B, what must be the acceleration of the car to cause A to start tipping? If the coefficient of friction between A and B is 0 5 and the acceleration of the car is gradually increased, will A slip before it tips?



562 The 320 lb door is suspended from shoes which rest upon a horizontal track, as in the cut The coefficient of friction between the shoes and track is 0 25 What force P is required to give the door an acceleration of 10 ft /sec²? What are the reactions of the shoes on the track? What will be the acceleration of the door after P is removed? *Ans* $P = 179$ lbs

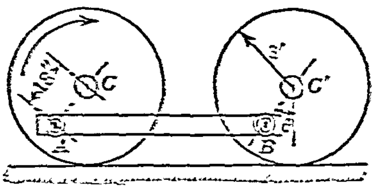


563. The straight post B rests upon the front edge of the car A, which is ascending the 30° incline with an acceleration of 20 ft /sec² At what angle θ must the post be inclined in order that it may maintain its position? *Ans* $\theta_x = 67° 35'$.

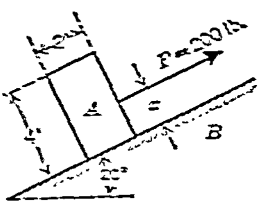


564. The driving wheels of a locomotive rotate at a speed of 360 r.p.m. The crank radius $r = 1.5$ ft.; the length of the side rod $l = 6$ ft.; the weight of the side rod is 150 lbs. Assuming the rod weight to be uniformly distributed along its length, find the inertia forces acting per unit length of the side rod.

Ans. 1660 lbs. per foot.



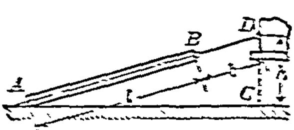
565. The side rod AB of a locomotive weighs 320 lbs. The speed of the locomotive at a certain instant is 60 m.p.h. and is decreasing at the rate of 5 ft./sec.². What are the pin reactions when $\theta = 60°$, when $\theta = 90°$, and when $\theta = 180°$? Draw free-body sketches showing the forces acting on the side rod.



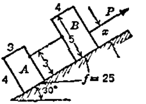
566. A homogeneous cylinder weighing 100 lbs. is being drawn up the 20° incline by the force P which acts parallel to the plane. The coefficient of friction between the cylinder and the plane is 0.2. Determine the limits of x, of the point of application of the force P, within which the cylinder will not tip over.

Ans. $x = 1.34$ ft. to 2.28 ft.

567. A timber of length l is dragged behind a truck. One end A slides over the surface of the road; the other end B is tied to a rope of length b which is attached to the truck at D. $DC = h$. The truck is moving with a uniform acceleration. Neglecting the cross-sectional dimensions of the timber, find the acceleration a of the truck when the timber and the rope form a straight line.



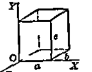
Ans. $a = (g/h)\sqrt{(l + b)^2 - h^2}$.



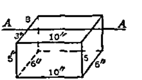
568 Blocks A and B are connected by a string, fastened in the position shown, and are drawn up the 30° incline by the force P. A weighs 160 lbs and B weighs 256 lbs. Determine the greatest force P and the limits (x) of its point of application, such that neither body A nor B will tip over.

Ans $P = 360.7$ lbs, $x = 1.08$ ft to 3.55 ft

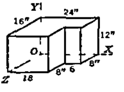
22 Moment and Product of Inertia.



569 Use the calculus to determine, directly, the moment of inertia of this homogeneous right parallelopiped with respect to the line OY. (Do not use the parallel axis theorem.) Show the dimensions of any elements chosen, and use limits of integration to apply to the axes here shown.



570 Determine the moment of inertia of the homogeneous rectangular parallelopiped with respect to the median line A–A of one face. The density of the material is 0.3 lb/cu in. *Ans* $I_A = 2.64$ lb in sec²



571 The material of the body shown weighs 0.3 lb per cubic inch. (*a*) Locate the center of gravity of the body. (*b*) Determine I_x, the moment of inertia about the x axis. (*c*) Determine the product of inertia P_{xz}.

Solution

(*a*) The body is symmetrical with respect to an xz plane that passes through the center of gravity. $\bar{y} = 6$ in. The x and z coordinates can be found by considering an area in the xz plane (§ 38).

$$x = \frac{8 \times 18 \times 9 + 8 \times 24 \times 12}{8 \times 18 + 8 \times 24} = 10.7 \text{ in}$$

$$z = \frac{16 \times 18 \times 8 + 6 \times 8 \times 4}{8 \times 18 + 8 \times 24} = 7.4 \text{ in}$$

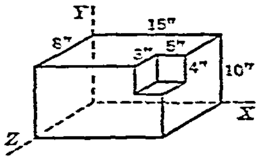
(*b*) Considering the body as two rectangular parallelopipeds (§ 93), we have

$$I_x = \frac{1}{3} \times \frac{16 \times 12 \times 18 \times 0.3}{386} \times (\overline{12}^2 + \overline{16}^2)$$
$$+ \frac{1}{3} \times \frac{6 \times 8 \times 12 \times 0.3}{386} (\overline{8}^2 + \overline{12}^2) = 357 + 31 = 388 \text{ lb in sec}^2$$

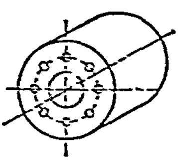
(c) The product of inertia is found by using the parallel-axes theorem (§ 98):

$$P_{zx} = \frac{16 \times 18 \times 12 \times 0.3}{386} \times (8 \times 9) + \frac{6 \times 8 \times 12 \times 0.3}{386} (21 \times 4)$$

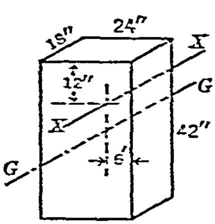
$$= 193.4 + 37.6 = \mathit{231\ lb.\ in.\ sec.^2}$$



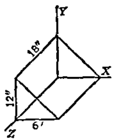
572. A rectangular parallelopiped, 15″ × 8″ × 10″, has one corner cut away as shown. The portion cut out is itself a rectangular parallelopiped, 5″ × 3″ × 4″. The material of the body weighs 0.283 lb./cu. in. (a) Locate the center of gravity. (b) Find the moment of inertia I_x about the x axis. (c) Find the product of inertia P_{zx}. *Ans.* $\bar{x} = 7.22$ in.; $\bar{y} = 4.84$ in.; $\bar{z} = 3.87$ in.; $I_x = 43.3$ lb. in. sec.^{2}; $P_{zx} = 22.8$ lb. in. sec.^{2}.



573. The hollow cylinder shown has an inner diameter of 12 inches, an outer diameter of 20 inches, and a length of 6 inches. The cylinder is made of wood weighing 45 lbs. per cu. ft. Eight holes are drilled entirely through the cylinder, in a direction parallel to the geometric axis of the cylinder. *Each hole is 2 inches in diameter, and its own axis is at a distance* of 8 inches from the axis of the cylinder. A solid steel pin is fitted into each hole, completely filling it. Calculate the moment of inertia of the entire body with respect to the axis of the cylinder. Steel weighs 490 lbs./cu. ft. *Ans.* $I = 12.08$ lb. in. sec.^{2}.

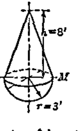


574. The moment of inertia of this rectangular parallelopiped with respect to the geometric axis GG is 17.75 lb. ft. sec.^{2}. The parallelopiped has the dimensions 18″ × 24″ × 42″ as shown, and weighs 40 lbs. per cu. ft. Determine the moment of inertia with respect to the XX axis indicated. (GG and XX are parallel to the 18-in. edge.) *Ans.* $I_x = 28.35$ lb. ft. sec.^{2}.



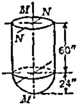
575. For the triangular wedge shown: (a) Compute the product of inertia P_{yz}. (b) Compute the moment of inertia I_z with respect to the z axis.

576. A body consists of a right circular cone made of wood (54 lbs./cu. ft.) and of a hemispherical base made of concrete (150



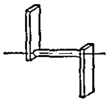
lbs./cu. ft.). Find the moments of inertia about the vertical axis and about a diameter MM.

Ans. $I_v = 1290$ lb. ft. sec.^{2}; $I_m = 2320$ lb. ft. sec.^{2}.



577. A body, of uniform density, consists of a solid right circular cylinder mounted on a solid hemispherical base. The weight of the body is 4 oz./cu. in. Find the moments of inertia about the geometric axis MM and about a diameter NN of the top of the cylinder.

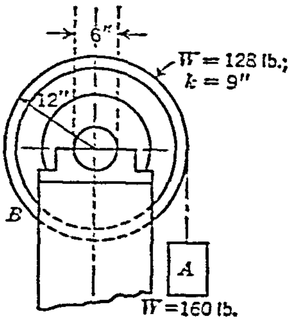
Ans. $I_m = 24{,}575$ lb. in. sec.^{2};
$I_n = 186{,}900$ lb. in. sec.^{2}.



578. Find the moment of inertia I about the axis of rotation and the product of inertia P_{xy} of the pedal shown in sketch.

579. Find the moment of inertia about the axis of rotation OZ and the product of inertia P_{yz} of the two balls carried by the bar CD of Problem 688.

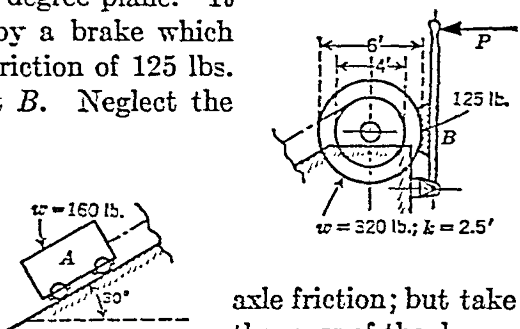
23. Rotation of a Rigid Body about a Fixed Axis.



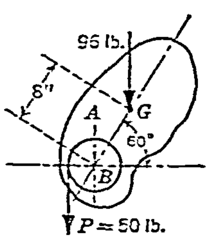
580. The body A descends 8 feet from rest and then strikes the ground. The axle diameter is 6 in. and the axle friction is 30 lbs. How many turns will B make after A stops? What is the tension in the cord before A stops?

Ans. 8.01 turns; $T = 55$ lbs.

581. The body A has an initial velocity of 20 ft. per sec. down a smooth 30-degree plane. It is stopped by a brake which develops a friction of 125 lbs. at the point B. Neglect the axle friction; but take into account the mass of the drum and the brake wheel. How far will the body A move down the plane? What is the tension in the cord while the body A is sliding down the plane with the brake set?



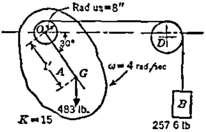
Ans. 38.2 ft.



582. The body A weighs 96 lbs. and has a cylindrical portion B whose radius is 3 in. The body rotates 8 in. out of center about a horizontal axis at B, normal to the plane in which the figure is shown. The body is also acted upon by a 50-lb. pull as shown. Its moment of inertia with respect to the axis through the center of gravity G is 2 lb. ft. sec.^{2}, and at the instant the body is in the position shown, the angular velocity is 4 rad./sec. Determine the angular acceleration and the axle reactions for the position shown.

Ans. $\alpha = 5.85$ rad./sec.; $R_x = -5.9$ lbs.; $R_y = 113$ lbs.

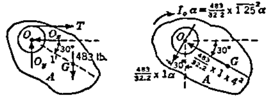
583. The body A rotates about a horizontal axis at O, and its mass center is at G. Its radius of gyration with respect to the axis of rotation is 15 in. Neglect the mass of D and neglect friction on the bearing at O. What are the horizontal and vertical



components of the axle reaction at O when the body is in the position shown? The angular velocity at the instant is 4 rad/sec. clockwise.

Solution:

The force system consisting of the forces acting on the body A shown in (a) is equivalent to the effective force system for A shown in (b) (§ 101).



(a) (b)

The acceleration of body B is $\frac{2}{3}\alpha$, where α is the angular acceleration of body A. The equation of motion for body B is

$$257.6 - T = \frac{257.6}{32.2} \times \frac{2}{3}\alpha.$$

For the body A, we have

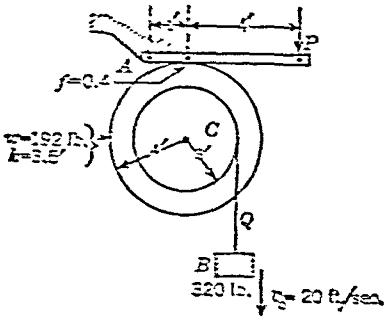
$$\Sigma M_o = \frac{2}{3}\ T + 483 \times 1 \times 0.866 = \frac{483}{32.2} \times \left(\frac{5}{4}\right)^2\alpha,$$

$$\frac{2}{3}\left(257.6 - \frac{257.6}{32.2} \times \frac{2}{3}\alpha\right) + 418 = 23.45\alpha,$$

$$171.8 + 418 = (3.55 + 23.45)\alpha, \quad \alpha = 21.8 \text{ rad/sec}^2, \quad T = 141.4.$$

Then

$$\Sigma F_y = O_y - 483 = 240 \times \tfrac{1}{2} - 15 \times 21.8 \times 0.866, \quad O_y = 318\ \textit{lbs},$$
$$\Sigma F_x = O_x - T = 15 \times 21.8 \times 0.5 + 240 \times 0.866,$$
$$O_x = 141.4 + 163.5 + 208 = 513\ \textit{lbs}$$



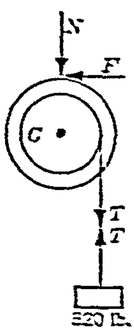
584. The motion of the drum C is controlled by the brake while the load B is lowered.

(a) What force P on the brake will stop B in 4 seconds?

(b) What is the tension in Q while B is being stopped?

(c) What constant acceleration will stop B in 4 seconds?

(d) How far does B travel in these 4 seconds?



Solution:

(a) The frictional force acting to stop the drum is expressed in terms of force P. Considering the equilibrium of the brake lever, we may write

$$\Sigma M_A = 5P - N = 0, \qquad N = 5P, \qquad F = 0.4N = 2P.$$

The drum is rotating about a fixed axis under the action of a constant torque (§ 101); hence we have

$$\Sigma M = I\alpha, \qquad T \times 3 - 4 \times 2P = \frac{192}{32.2} \times \overline{3.5}^2\alpha.$$

For the body B, we have

$$320 - T = \frac{320}{32.2}a, \qquad \text{and} \qquad a = 3\alpha.$$

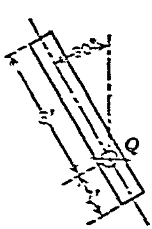
To stop B in 4 seconds, we must have

$$v_B = 20 + a(4) = 0, \qquad a = -5 \text{ ft./sec.}^2 \text{ (upward)}, \qquad \alpha = -5/3 \text{ rad./sec.}^2$$

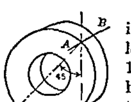
From the above equation, we find

$$\begin{aligned} T - \tfrac{8}{3}P &= 24.3(-5/3), \\ 320 - T &= 9.93(-5), \\ \hline 320 - \tfrac{8}{3}P &= -90.15, \\ P &= 154 \text{ lbs.} \end{aligned}$$

585. This block is 4 ft. long, 6 in. wide, and 3 in. thick, and weighs 352 lbs. It rotates in a vertical plane about a horizontal fixed shaft at Q. At the instant it is in the position shown, it has a speed of 60 r.p.m. and is changing its angular velocity due to the action of gravity. Neglecting axle friction, determine the components of the axle reaction.

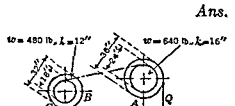


Ans. $R_x = 152$ lbs.; $R_y = -59$ lbs.



586. The outer diameter of this hollow cylinder is 8 ft., the inner diameter is 3 ft., and its length is 1 ft. The weight of the material is 100 lbs./ft.^{3}. The cylinder is rotated about a horizontal axis AB. Its speed when in the position shown is 40 r.p.m. Determine the axle reactions.

Ans. $R_x = 8080$ lbs.; $R_y = 9630$ lbs.

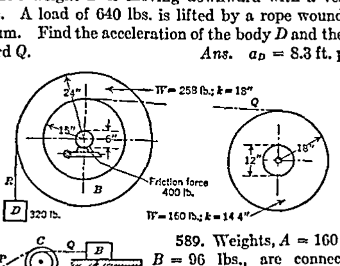


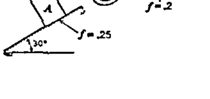
587. The weight D, which is being lowered, has a velocity of 10 ft./sec. when the foot brake is applied. The force P is 200 lbs. and friction at C has a coefficient of ½. Use the acceleration method to determine whether the system will stop. What is the tension in the cord Q after the brake is applied? How far will body D move during the first two seconds after the brake is applied?

Ans. $T_Q = 1041$ lbs.; $s = 17$ ft. down.

588. The weight D is moving downward with a velocity of 20 ft./sec. A load of 640 lbs. is lifted by a rope wound on the 12-in. drum. Find the acceleration of the body D and the tension in the cord Q.

Ans. $a_D = 8.3$ ft. per sec.^{2}.

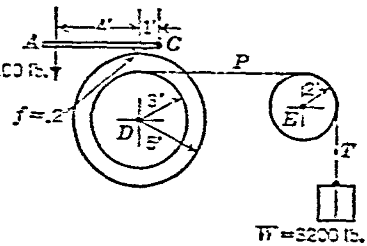




589. Weights, $A = 160$ lbs. and $B = 96$ lbs., are connected by ropes wound on the step pulley C. The diameter of the larger pulley at C is 4 ft. and of the smaller is 2 ft. The weight of the two pul-

leys combined is 218 lbs. The radius of gyration of the two pulleys combined is 1.5 ft. Determine the acceleration of A and the tensions in the two cords P and Q.

Ans. $a_A = 3.76$ ft. per sec.^{2}; $P = 26.8$ lbs.; $Q = 24.8$ lbs.



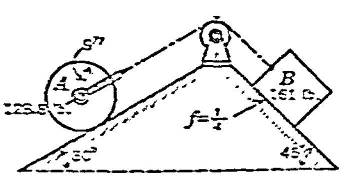
590. The foot brake AC is applied to the drum D when a weight W is being raised at a speed of 20 ft./sec. D weighs 960 lbs., its two diameters are 6 ft. and 10 ft., as shown, and its radius of gyration is 4 ft. E weighs 192 lbs., its diameter is 4 ft., and its radius of gyration is 1.5 ft. Determine the retardation of W and the tensions P and T in the rope. How much farther will W rise before stopping? Neglect axle friction.

Ans. $a = 21.6$ ft. per sec.^{2}; $T = 1050$ lbs.; $P = 978$ lbs.

24. Plane Motion of a Rigid Body.

591. A homogeneous cylinder which is 3 ft. in diameter and which weighs 644 lbs. rolls down an inclined plane that makes an angle θ with the horizontal. If the coefficient of friction $f = 0.3$, find the maximum angle θ for which the cylinder will roll without slipping. Compute the acceleration of the center O and all forces acting on the cylinder.

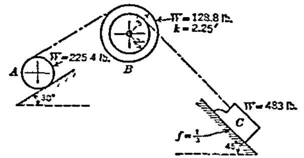
Ans. $\theta = 42°$. $a = 14.3$ ft. per sec.^{2}.



592. A solid spherical body A having a radius of 9 in. and a weight of 128.8 lbs. is connected to a second body B by a cord which passes over a smooth peg. The cord is fastened to an axis through the center of the sphere which rolls on the plane. Body B, weighing 161 lbs., slides on the inclined plane. If at a certain instant the body B is moving up the plane with a velocity of 10 ft. per second, find the acceleration of B and the tension in the cord. *Ans.* $a = 7.4$ ft. per sec.^{2}; $T = 105$ lbs.

593. At a certain instant body C is moving up the plane with a velocity of 20 ft. per sec. The cylinder A rolls on the plane

without slipping, and as it rolls the rope is wound onto the cylinder. (*a*) Determine how far the center of A will move

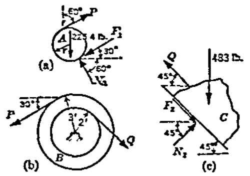


before it stops. (*b*) What is the tension in the cord AB during this time?

Solution:

The equations of motion for the bodies A, B, and C are written with reference to the free body diagrams (a), (b) and (c) (§ 106*a*). For A, we have

$$P - F_1 - 112.7 = \frac{225.4}{32.2}\,a_A, \qquad Pr + F_1 r = \frac{225.4}{32.2}\,\frac{r^2}{2}\,\alpha_A.$$



For B:

$$2Q - 3P = \frac{128.8}{32.2}\,(2.25)^2\,\alpha_B.$$

For C:

$$341 + F_2 - Q = \frac{483}{32.2}\,a_C, \qquad N_2 - 483 \times 0.707 = 0,$$

$$N_2 = 341 \text{ lbs.}, \qquad F_2 = 114 \text{ lbs.}$$

The relations between the accelerations are

$$a_A = r\alpha_A, \quad 2r\alpha_A = 3\alpha_B, \quad 2\alpha_B = a_C, \quad \alpha_B = \tfrac{2}{3}\,a_A, \quad a_C = 4/3\,a_A.$$

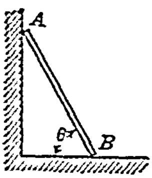
Solving the above equations, we find

$$P - F_1 - 112.7 = 7a_A,$$
$$P + F_1 = 3.5a_A,$$
$$4/3\,Q - 2P = 4 \times \tfrac{2}{3} \times (2.25)^2 \times 2/3\, a_A = 9a_A,$$
$$455 \times 4/3 - 4/3\,Q = 20 \times 4/3\, a_A.$$

Adding, we get

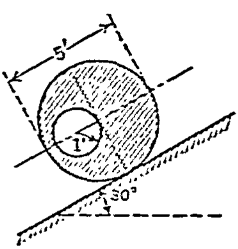
$$607 - 112.7 = 46.15a_A, \qquad a_A = 10.7 \text{ ft./sec.}^2.$$

The distance traveled by center of body A is $s = $ *10.5 ft.* The tension is $P = $ *113 lbs.*



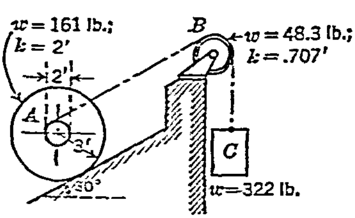
594. A straight uniform bar AB slides down with its upper end against a smooth vertical wall and its lower end on a smooth horizontal floor. The bar is 8 ft. long and weighs 80.5 lbs. It is required to determine the angular acceleration of the bar when $\theta = 60°$. Also determine the forces acting on the bar at A and B.

Ans. $\alpha = 3.02$ rad. per sec.^{2}; $R_A = 26.1$ lbs.; $R_B = 65.4$ lbs.



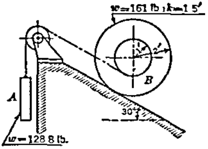
595. A cylinder 5 ft. in diameter with a hole through it as shown, rolls without slipping on the inclined plane. The body weighs 644 lbs. When the body is in the position shown, the velocity of the center is 5 ft. per sec. down the plane. Calculate the angular acceleration and all the forces acting on the body.

Ans. $\alpha = 3.6$ rad. per sec.^{2}; $F = 126$ lbs.; $N = 572$ lbs.



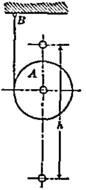
596. A wheel and drum weighing 161 lbs. are rigidly fastened together as shown for body A in the diagram. The rope is wound around the drum of A and over a pulley B, whose diameter is 2 ft. A is supported on a smooth inclined plane. Determine the acceleration of the body C. The cord between A and B makes an angle of 30° with the horizontal.

Ans. 17.6 ft. per sec.^{2}.



597. A wheel and drum rigidly fastened together have a total weight of 161 lbs. Their radius of gyration with respect to a gravity axis at right angles to the plane of motion is 1.5 ft. The wheel rolls without slipping. A cord which wraps around the drum passes over a small pulley and supports a weight of 128.8 lbs. Determine the acceleration of the body B and all the forces acting on B.

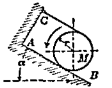
Ans. $\alpha_B = 0.91$ rad. per sec.^{2}; $T = 132$ lbs.; $F = 61.1$ lbs.



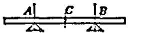
598. A *circular cylinder* A *of weight* w has a thread wound around its middle. One end of the thread is fixed at B. The cylinder is dropped and falls vertically, unwinding the thread. Find the velocity v of the cylinder's axis after it has descended a distance h. Find the tension T in the thread.

Ans. $v = (2/3)\sqrt{3gh}$; $T = w/3$.

599. A solid cylinder M of weight P and radius r has two strings wound around it. The windings are symmetrical with respect to the middle plane of the cylinder and they are parallel to the bases. The cylinder is placed on an inclined plane AB and the strings are tied to a rod C at a distance $AC = 2r$ from the surface of AB. The coefficient of friction between the cylinder and the plane is f. The cylinder starts from rest and moves down the plane under the action of gravity. Find the distance S through which the center of gravity of the cylinder moves in time t and the tension T in the strings.



Ans. $S = \frac{1}{3}g(\sin\alpha - 2f\cos\alpha)t^2$; $T = \frac{1}{6}P(\sin\alpha + f\cos\alpha)$.



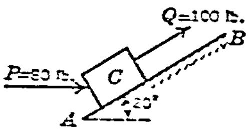
600. A thin rod of length $2l$ and weight P lies on two supports A and B. The center of gravity C is equidistant from both

supports. $CA = CB = a$. The force on each support is $\frac{1}{2}P$. How will the force on A change when B is suddenly removed?

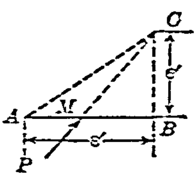
Ans. $R_A = P\dfrac{l^2}{l^2 + 3a^2}$.

WORK AND ENERGY

25. Work and Energy.

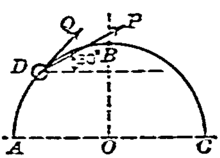


601. A body C is moved up the plane by the horizontal force P and the force Q. The frictional resistance is 10 lbs. C weighs 40 lbs. Compute the work done on C by each force acting during a displacement, from A to B, of 20 ft. What is the total work done? *Ans.* $+$ 3030 ft. lbs.



602. A body M is caused to move from A to B by the action of several forces. One of these forces P, whose magnitude is 10 lbs., is always directed toward C. How much work does this force do while M moves from A to B?

Ans. $+$ 40 ft. lbs.



603. ABC is a smooth rail in the form of a vertical semicircle of 4 ft. radius; D is a body weighing 50 lbs. which can be made to slide along the rail. P is a force of 150 lbs., always inclined 30° to the horizontal; Q is a force of 40 lbs., always directed along the tangent. Compute the work done on D by all the forces acting on it while D is moved from A to B.

Solution:

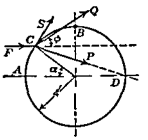
The work done by each force is $W = \int F \cos\theta \, ds$ (§ 110).

$$W_Q = \int_0^{2\pi} 40 \times \cos 0 \times ds = 40 \times 2\pi = \mathit{251\ ft.\ lbs.}$$

$$W_P = \int_0^{2\pi} 150 \cos\left(90° - \frac{s}{4} - 30°\right) ds$$

$$= 150 \int_0^{2\pi} \cos\left(60° - \frac{s}{4}\right) ds = \mathit{820\ ft.\ lbs.}$$

$$W_G = -50 \times 4 = -\mathit{200\ ft.\ lbs.}\ (\S\ 110b).$$



604 C is a bead on a circular wire ABD and is subjected to four forces F, P, Q, and S $F = 10$ lbs and is always horizontal $Q = 100$ lbs and acts at an angle ϕ to the horizontal which is always equal to angle α S is a tangential force and its value in pounds is $40s^2$, where s is the arc AC in feet P is 40 lbs and is always directed toward D Compute the amount of work done by each force for the displacement of C from A to B

Ans $W_F = 40$ ft lbs, $W_Q = 400$ ft lbs, $W_S = 3307$ ft lbs, $W_P = 94$ ft lbs

605 A train weighing 1,600,000 lbs is moving with a velocity of 45 ft /sec when the engineer shuts off the steam The train, decelerating uniformly under the action of friction, coasts 6000 ft and comes to a velocity of 6 ft /sec Find the energy in ft -lbs lost in frictional work up to this point and the time it takes to coast the 6000 ft

Ans $E = 49{,}500{,}000$ ft lbs, $t = 235$ seconds

606 A shell weighing 12 lbs leaves the muzzle of a gun with a velocity of 1710 ft /sec, having traveled 6 ft inside the barrel Find the average force P of the powder gases while the shell is moving through the gun Assuming the gas pressure constant, find the time necessary for the shell to travel through the gun barrel What resisting force would be necessary to stop the shell in a distance of 0 3 ft ?

Ans (1) $P = 91{,}000$ lbs, $t = 0\ 0070$ sec (2) 1,820,000 lbs

607 Find the horsepower of an engine which lifts a hammer weighing 440 lbs to a height of 2½ ft 120 times per minute

Solution

The work done by the machine in one minute is (§ 110a)

$$W = 120 \times (2\tfrac{1}{2} \times 440) = 132\ 000 \text{ ft lbs}$$

The machine develops 132 000 ft lbs /min or

$$\frac{132\ 000}{33\ 000} = 4\ H\ P$$

(§ 114 table of units)

608 Niagara Falls is 200 ft high and 286,000 cu ft of water flow over it every second Imatra Falls is 40 ft high and 13,000

cu. ft. of water flow over it every second. Compute the horse power of these two falls.

Ans. Niagara: 6,500,000 H.P.; Imatra: 59,000 H.P.

609. The central station of a car line supplies power to 45 cars each weighing 20,000 lbs. The frictional resistance is 2% of the car weight and the average velocity of each car is 10 mi./hr. Find the horse power developed by the central station.

Ans. 480 H.P.

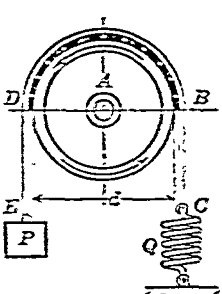
610. A pump driven by a 2 H.P. motor lifts 150,000 cu. ft. of water 10 ft. The overall efficiency of the installation is 80%. How long does it take to pump the water? *Ans.* 29 hrs., 30 min.

611. A coal barge is unloaded by means of a bucket weighing 1000 lbs. and having a capacity of 1000 lbs. The barge contains 3,600,000 lbs. of coal and must be unloaded in 10 hrs. The coal is lifted 27.5 ft. by the bucket. What is the effective horse power of the motor driving the bucket? *Ans.* 10 H.P.

612. A weight of 40 lbs. is pulled up an inclined plane through a distance of 18 ft. The angle between the plane and the horizontal is 30° and the coefficient of friction is 0.01. Calculate the work done by the friction force and the weight. *Ans.* 366.2 ft. lbs.

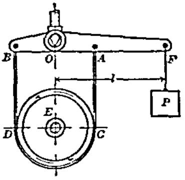
613. The engine of a steamer traveling at a speed of 16.5 knots develops 5064 indicated horse power. The overall efficiency of the engine and propeller is 40% (1 knot = 1.689 ft./sec.). Find the resistance to the motion of the steamer. *Ans.* 40,000 lbs.

614. A single-acting steam engine has a mean effective steam pressure of 66 lbs./sq. in., a piston area of 50 sq. in., and a stroke of 15 in. It runs at 120 r.p.m. and has an efficiency of 90%. What is its horse power? *Ans.* 13.5 H.P.

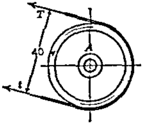


615. To determine the power of a motor, a pulley A, $28\frac{7}{8}$ in. diameter, is mounted on its shaft. A band of wooden blocks fits over the pulley. Its right end BC is held by a spring scale Q; and a load $P = 2$ lbs. is fixed to the left end DE. At 120 r.p.m. the spring scale shows a tension of 10 lbs. What horse power is the motor developing?

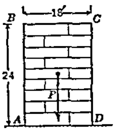
Ans. 0.22 H.P.



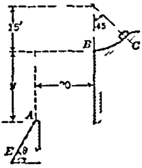
616 A dynamometer used to measure the power of motors consists of a band $ACDB$ passing over a pulley E on the motor shaft and attached to the lever BF, which rests on a support at O By adjusting the support the band tension can be varied The lever BF is kept horizontal by means of the weight P at F When the motor is running at 240 r p m , $P = 6$ lbs and its distance from O is $l = 22$ in Find the horse power being developed by the motor *Ans* 0 50 H P



617 A belt transmits 20 H P to the pulley A, which is 40 in in diameter and is running at 150 r p m The tension T in the tight side of the belt is twice the tension t in the slack side Find the values of T and t *Ans* $T = 840$ lbs , $t = 420$ lbs



618 A solid block of masonry with the dimensions shown in the sketch weighs $P = 8000$ lbs Find the work done in tipping the block over the edge D *Ans* 24,000 ft lbs



619 It is desired to design a package chute as shown here A package is placed in a practically frictionless circular guide at C and allowed to slide due to the action of gravity It is projected into space as it leaves the guide at B, jumping a horizontal distance of 20 ft before striking a ramp at A Use the principle of work and energy to determine the velocity of the package when it leaves the guide Find the vertical distance y to A, where the package will strike the ramp Determine what the angle of inclination of AE must be in order

that it may be tangent to the path of the center of gravity of the package at the instant it touches the ramp.

Ans. $v = 16.8$ ft./sec.; $y = 22.7$ ft.; $\theta = 66° 15'$.

620. A train weighing 2,000,000 lbs. approaches a station which is on a 0.4% grade. At a distance of 1500 ft. from the station the engineer applies the brakes. At the time the brakes are applied, the train is running at a speed of 36 ft. per sec. up the grade. The frictional resisting force of the train is 4000 lbs. Find the additional braking friction force necessary to stop the train at the station.

Solution:

The change E in kinetic energy of the train is equal to the work W done by all forces acting on the train (§ 119).

$$E = \frac{2 \times 10^6}{2 \times 32.2}(0 - 36^2);$$

$$W = -2 \times 10^6 \times 0.004 \times 1500 - 4000 \times 1500 - F \times 1500,$$

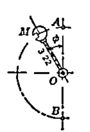
where F is the additional braking friction force. Equating E and W, we find $F = 14,800$ *lbs.*

621. A train weighing 500,000 lbs. runs on a horizontal track with an acceleration of 0.644 ft./sec.^{2}. The frictional resistance of the train is 1% of its weight. At a certain time the train has a speed of 48.56 ft. per sec. Find the horse power being developed by the locomotive 10 sec. later. *Ans.* 1500 H.P.

622. A ram is used to pack down earth. It weighs 120 lbs., has a cross-sectional area of 12 sq. in. and falls through a height of 30 in. At the last blow the ram sinks ½ in. into the ground. Assuming that the resisting force of the ground remains constant during the penetration of the ram and that it can hold any load not exceeding the value of this force, find the maximum load under which the ground will not settle. *Ans.* 600 lbs./sq. in.

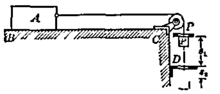
623. A small box-car weighing 12,000 lbs. offers a frictional resistance to motion of 30 lbs. A workman exerting a push of 50 lbs. starts the car moving over a straight horizontal track. He pushes it 60 ft. and then lets it go. Neglecting air resistance, find the highest speed v_{max} attained by the box-car and the total distance s which the car moves before stopping.

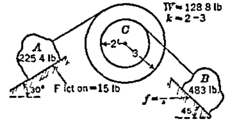
Ans. $v_{max} = 2.54$ ft./sec.; $s = 100$ ft.



624 The hammer M of an impact testing machine is attached to the end of a light rod OM, 3.22 ft long, which is pivoted at O. The hammer leaves the point A with a negligibly small velocity. Neglecting the weight of the rod and the small pivot friction and considering the hammer M as a mass point, find the velocity v with which it passes through the lowest point B. *Ans* $v = 20.4$ ft/sec

625 A weight p carrying a washer p_1 is suspended on a string which passes over a pulley and is attached to a block A. The weight of the block is Q. Under the action of the weights p and p_1 the block begins to slide across the rough horizontal table BC. After dropping a distance s_1 the weight p passes through a ring D which catches the washer p_1. After dropping a further distance s_2 the weight p comes to rest. Neglecting the effects of the weight of the string and of the pulley friction, find the coefficient of friction f between the block and the table. Take $Q = 1.6$ lbs, $p = 0.2$ lbs, $p_1 = 0.2$ lbs, $s_1 = 20$ in, $s_2 = 12$ in. *Ans* $f = 0.2$





626 At a given instant, the body A has a downward velocity of 20 ft per sec parallel to the plane. (a) How far will A move before stopping? (b) What is the tension in the cord BC? (c) At the initial instant, what power is developed by all the forces acting on body B? Use the principle of work and kinetic energy.

Solution

The work done by all the forces acting on bodies A, B, and C is equal to the change in kinetic energy for the system (§ 119a).

If the body A moves a distance s the body C turns through an angle $s/3$ and B moves a distance $\frac{2}{3}s$. The work done is given by the equations

$$W_A = 225.4 \sin 30° \times s - 15 \times s = 97.7\,s$$
$$W_B = -114 \times \tfrac{2}{3}s - 341 \times \tfrac{2}{3}s = -303\,s$$
$$\text{Total Work} = -205.3\,s$$

The change in kinetic energy is:

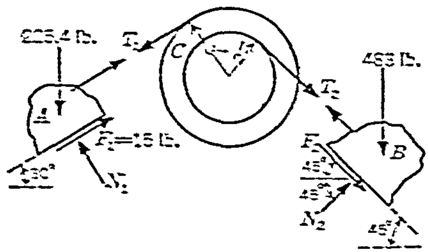
$$\Delta KE_A = \frac{1}{2} \times \frac{225.4}{32.2}(0 - \overline{20}^2) = -1400$$

$$\Delta KE_C = \frac{1}{2} \times \frac{128.8}{32.2} \times \overline{2.25}^2\left(0 - \frac{400}{9}\right) = -\ 450$$

$$\Delta KE_B = \frac{1}{2} \times \frac{483}{32.2} \times \left(0 - \frac{400 \times 4}{9}\right) = -1333$$

$$\text{Total change in } KE = \overline{-3183} \text{ ft. lbs.}$$

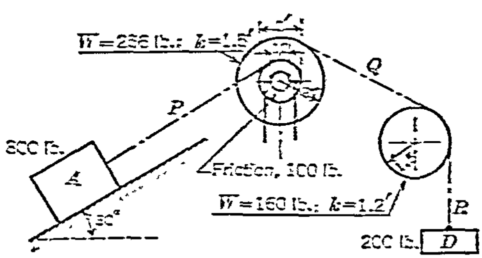
$$-205.3s = -3183, \quad s = 15.5 \text{ ft.}$$



Considering the work done by all forces acting on body B alone and the change in KE of B, we find

$$(T_2 - 114 - 341)\tfrac{2}{3} \times 15.5 = -1333, \qquad T_2 = 326 \text{ lbs.}$$

627. At a given instant, the body A has a velocity of 8 ft./sec. up the plane. Use the principle of work and kinetic energy to determine how far the body A will move before its velocity is



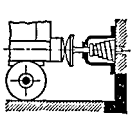
reduced to 5 ft./sec. Use the same principle to find the tension in the cord Q during this time. *Ans.* $s = 8.4$ ft.; $T = 244$ lbs.

628. The deflection x of a spring is proportional to the pull exerted on it, the constant of proportionality being 2000 lbs. per

inch. Give the potential energy of the spring as a function of x. *Ans.* $V = 1000x^2$ in. lbs.

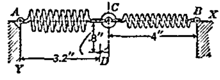
629. The spring of a spring-gun has a free length of 8 in. The spring characteristic is 1 lb. per in. The spring is compressed to a length of 4 in. and a ball weighing 1 ounce is put in the barrel of the gun against the compressed spring. Find the velocity v with which the ball will leave the gun. *Ans.* $v = 26.2$ ft./sec.

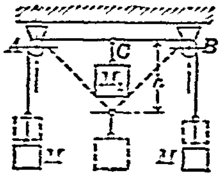
630. The static deflection of a beam loaded in the middle by a weight $Q = 4000$ lbs. is 0.08 in. Find the maximum deflection of the beam when the weight Q is placed just above the middle of the undeflected beam and released. Find the maximum deflection when the weight Q is dropped on the middle of the beam from a height of 4 in. *Ans.* (1) $x = 0.16$ in.; (2) $x = 0.884$ in.



631. A box-car weighing 32,000 lbs. runs into a spring buffer at the end of a siding with a velocity of 3 ft./sec. The characteristic of the buffer spring is 25,000 lbs. per inch. Find the maximum compression of the spring after the impact. *Ans.* $x = 2.07$ in.

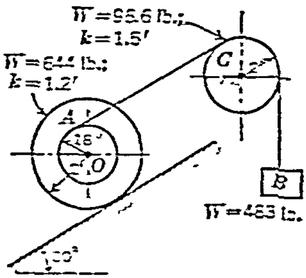
632. Two unstrained springs AC and BC attached to the points A and B lie along the horizontal line AX. The two free ends of the springs are attached to the body C, which weighs 3.22 lbs. The spring characteristics are 10 lbs. per in. for AC and 20 lbs. per in. for CB. $AC = BC = 4$ in. The body is given a blow which imparts to it a velocity $v_0 = 6$ ft./sec. It moves in such a way that it subsequently goes through the point D located at $x_0 = 3.2$ in., $y_0 = 0.8$ in. A is the origin of the coordinate system as shown in the sketch. Find the velocity of the body as it passes through D. *Ans.* $v_d = 4.38$ ft. per sec.





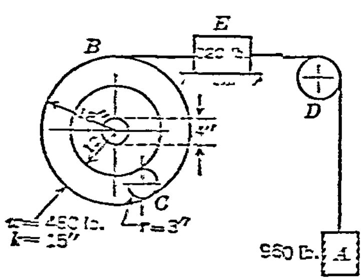
633. Two equal weights M, p lbs. each, are suspended on a rope passing over two very small pulleys A and B. $AB = 2l$. At the point C, half way between A and B, a load M_1 is suspended. It weighs p_1 lbs. The weight M_1 is dropped without initial velocity. Find the maximum distance h which the weight M_1 will fall.

$$\textit{Ans.}\quad h = \frac{4pp_1l}{4p^2 - p_1^2}.$$



634. A wheel and drum rigidly fastened together have a total weight of 644 lbs. Their radius of gyration with respect to the axis through the centroid O perpendicular to the plane of motion is 14.4 in. The wheel rolls without slipping and, as it rolls, a cable is unwound from the drum and passes over the pulley C. The pulley has a weight of 96.6 lbs. and a radius of gyration of 1.5 ft. A weight B of 483 lbs. is attached to the lower end of the cable. Calculate the velocity of the center of the wheel at the instant when the weight B has descended 15 ft., starting from rest. Use the principle of work and energy. *Ans.* $v = 10.6$ ft. per sec.

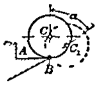
635. The motor C is geared to the drum B which draws the weight E over a smooth plane, and raises the weight A, by means of a cord passing over a fixed pulley D. The axle of drum B is 4 inches in diameter and the friction on it is 200 lbs. Neglect the mass of D and its axle friction, as well as the kinetic energy of the rotating parts of the motor. Use the principles of work and energy to find the distance which A will be moved, starting from rest, before it acquires a velocity of 10 ft./sec., if the torque of the motor is constantly 412 pound-feet. Determine the angular velocity of the motor shaft when it is exerting a torque of 412 lbs.-ft. and is delivering 20 H.P.



Ans. (1) $y = 19.7$ ft.; (2) 26.7 rad. per sec.

636 Find the ratio of the height h of a solid circular cylinder to its radius R, for which the kinetic energy about any axis through its center of gravity will be the same with the same angular velocity *Ans* $h/R = \sqrt{3}$

637. A solid cylinder of weight Q and radius r rests on a horizontal shelf, AB It is given a negligible initial velocity and rolls over the sharp edge B without sliding At the instant the cylinder leaves the shelf the plane through the axis of the cylinder and the edge of the shelf forms an angle $CBC_1 = \alpha$ with the vertical Find the angular velocity ω of the cylinder after it leaves the shelf, and the angle α Neglect the effects of rolling friction and of air resistance



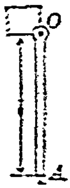
Ans $\omega = 2\sqrt{\frac{g}{7r}}$, $\alpha = 55° \, 10'$

638 The rotating part of a turbogenerator weighs 240,000 lbs and its radius of gyration is 20 in The rotor runs at 1800 r p m, and the steam develops 46,000 H P Due to a short circuit on the line, the circuit breaker opens, and the resistance to the rotor's rotation is thus suddenly removed The governor closes the inlet valve in 2 5 seconds Assuming that until that instant the steam exerts the same pressure on the turbine blades, what is the speed of the rotor at the instant of valve closure?

Ans 1940 r p m

639 A turning moment of 3000 ft -lbs acts on a flywheel which weighs 16,100 lbs and has a radius of gyration of 3 ft The flywheel starts from rest How long will it take to reach a speed of 120 r p m ? How much work is expended in bringing it to this speed? *Ans* 18 8 sec , 355,000 ft lbs

640 A stone parallelopiped 12 ft long, 6 ft wide, and 4 ft thick, weighing 322 lbs per cu ft , rotates about an axis which passes through one of its diagonals It starts from rest and reaches an angular velocity of $\omega = 7$ rad per sec Find the work expended *Ans* 483,000 ft lbs.



641. A rod OA of length $l = 10.73$ ft. is hinged at its upper end O and hangs vertically when at rest. It is struck a transverse blow and rotates through an angle of 90°. Find the initial velocity given to end A.

Ans. $v_A = 32.2$ ft./sec.

642. A shell 4 in. in diameter, weighing 36 lbs., travels with a velocity of 1500 ft. per sec. and at the same time rotates about its axis at a speed of 100 revolutions per sec. Considering the shell as a uniform cylindrical body, find its kinetic energy.

Ans. $E = 1{,}261{,}000$ ft.-lbs.

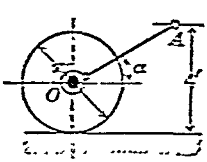
643. A shaft of 4 in. diameter coasts down from 60 r.p.m. The coefficient of friction in the journals is 0.05. How many revolutions will it make before it comes to rest?

Ans. $n = 0.16$ rev.

644. A shaft of 4 in. diameter weighing 1000 lbs. has a fly-wheel mounted on it which is 6 ft. in diameter and weighs 6000 lbs. The shaft is rotating at a speed of 60 r.p.m. when the driving power is shut off. The coefficient of friction in the bearings is 0.05. How many revolutions will the shaft make before it comes to rest?

NOTE: The weight of the flywheel can be assumed to be concentrated in the rim.

Ans. $n = 90.5$ rev.

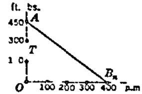


645. A roller 4 ft. in diameter and weighing 1120 lbs. is pushed by a man exerting a constant force P along the line OA. The handle OA is 5 ft. long; the hand-grip at A is 4 ft. above the floor. The man starts the roller from rest and brings it to a velocity of 4 ft. per sec., having moved it a distance of 7 ft. Neglecting the effects of friction, find the force P.

Ans. $P = 65$ lbs.

646. Water enters a hydraulic turbine with a velocity of 12 ft. per sec. and leaves it at a velocity of 3 ft. per sec. The difference in level between inlet and outlet is 4½ ft. 600 lbs. of water flows through the turbine each second. The efficiency of the turbine is 65%. Find the power output of the turbine.

Ans. 4.68 H.P.



647 The speed torque curve of a certain hydraulic turbine is a straight line AB, as shown in the sketch Find the power P of the turbine as a function of the speed Give its maximum value

Ans $P = 0\,000213n(400 - n)$ H P, $P_{max} = 8\,57$ H P

648 A train is running at a uniform speed of 36 miles per hour on a straight and level track To increase the speed of the train, the engineer opens the throttle, increasing the drawbar pull of the locomotive 25% The frictional resistance of the train is 1/200 of its weight and it is independent of the speed How many miles will the train run before its speed is increased to 45 miles per hour? *Ans* 3 7 mi

649 Two particles are charged with positive electricity The charge on the first particle is $q_1 = 100$ absolute electrostatic units (C G S) and the charge on the second is $q_2 = 0\,1q_1$ The first particle is fixed and the second particle is moving away from the first under the action of the repulsive force $F = q_1q_2/r^2$ dynes, where r is the distance in cm between the particles At time $t = 0$, $r = 5$ cm and the velocity of the moving particle is zero The mass of the moving particle is 1 gram Find the maximum velocity which the moving particle can attain

Ans $V_{max} = 20$ cm per sec, at r = infinity

650 The gravitational force at any point inside the earth is proportional to the distance r from the center of the earth and is directed toward the center The diameter of the earth is $41\,85 \times 10^6$ ft and the acceleration of gravity at the surface is $32\,2$ ft/sec^2 Assume that a diametral shaft could be drilled through the earth If a body were dropped into the shaft at the surface of the earth, what velocity would it have when it passed through the center of the earth? *Ans* 4 91 mi/sec

651 A body is projected vertically upward from the surface of the earth The force of gravity acting on it is inversely proportional to the square of the distance to the center of the earth The radius of the earth is $20\,92 \times 10^6$ ft and the acceleration of gravity is $32\,2$ ft/sec^2 at the surface of the earth Neglecting

the effects of air resistance, find the initial velocity of the body if it reaches a height equal to the radius of the earth.

Solution:

The body, whose weight is P, moves under the action of a downward force $F = P(R^2/x^2)$, where x is the distance of the body from the center and R is the radius of the earth.

At the highest point, the velocity of the body is zero; the change in kinetic energy of the body is equal to the work done on the body (§ 118); hence

$$\frac{1}{2}\frac{P}{g}(0 - v_0^2) = \int_R^{2R}(-F)dx = -\int_R^{2R} P\frac{R^2}{x^2}dx = -\frac{PR}{2},$$

where v_0 is the initial velocity of the body.

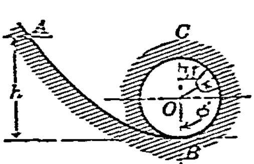
$$v_0 = \sqrt{gR} = 26{,}000 \text{ ft./sec., } (= 4.91 \text{ mi./sec.}).$$

652. A shell is shot from the earth in the direction of the moon. It reaches the point where the attractive forces of the moon and the earth are equal and remains there. Neglecting the influence of the motion of the earth and that of the moon, find the initial velocity v_0 of the shell. The radius of the earth is $R = 4000$ mi.; the distance between the centers of the earth and the moon is $d = 60R$. The ratio of the masses of the moon and the earth is $1 : 80$. The acceleration of gravity is 32.2 ft./sec.^{2} at the surface of the earth. *Ans.* 6.9 mi./sec.

653. A 2-lb. weight is suspended on a string 20 in. long, the other end of which is fixed. The pendulum is displaced an angle of 60° from its vertical position and the weight is given a velocity $v_0 = 80$ in. per sec. The velocity v_0 is in the vertical plane; it is normal to the string and is directed downward. Find the tension in the string when the weight passes through its lowest position. Find the elevation above this point that the weight will reach.

Ans. 5.66 lbs.; the elevation is 18.3 in.

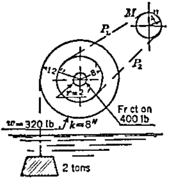
654. In the previous problem, find the value of the initial velocity v_0 for which the weight will travel around the whole circle. *Ans.* $v_0 \geqq 152$ in./sec.



655. The car of a loop-the-loop weighs p lbs. It gathers momentum by running down the inclined track AB and then it runs around the inside of the circular loop CB of radius a ft. Find the height h from

which it is necessary to start the car in order that it can run around the loop without leaving the rails Find the force N against the rails at M, where angle $MOB = \phi$

Ans $N = p\left[\frac{2h}{a} - 2 + 3\cos\phi\right]$, $h = 2.5a$



656 An electric motor M is used to lift a stone weighing 2 tons when submerged in water, from a depth of 120 ft, and it is found that the velocity at which the stone emerges from the surface is 20 ft/sec The resistance offered by the water to the motion of the stone is constant and is 20 per cent of its weight in water The force exerted by friction upon the axle of the hoisting drum is 400 lbs Find the constant torque exerted by the motor, the power output in H P of the motor when the block reaches the surface, and the difference in belt tensions on the two sides of the motor pulley

Ans $T = 2605$ lbs-ft, H P $= 189.5$, $P_1 - P_2 = 7815$ lbs

IMPULSE AND MOMENTUM

26 Impulse and Momentum

657. A body weighing 3 lbs moved to the left with a velocity 15 ft per sec A force, directed to the right, was applied to it for 30 sec, after which the velocity of the body was 165 ft per sec to the right Find the force and the work done by it

Solution

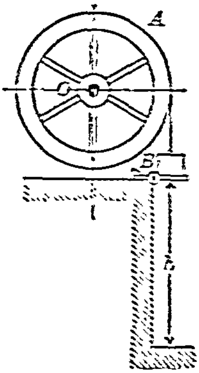
The impulse imparted to the body toward the right equals the change in momentum in that direction (§ 122), hence

$$F \times 30 = \frac{3}{32.2}[165 - (-15)] \qquad F = 0.56\ lb$$

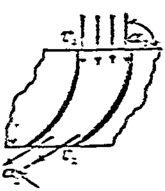
Using the principle of work and kinetic energy (§ 119a), we see that the distance traveled is given by the equation

$$0.56 \times s = \frac{1}{2} \times \frac{3}{32.2}(\overline{165}^2 - \overline{15}^2), \qquad s = 2240\ ft.$$

The work done is $W = 0.56 \times 2240 = 1257$ ft lbs.



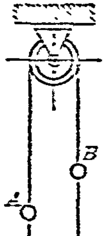
658. The flywheel A has a radius $R = 20$ in. In order to find its moment of inertia I about the axis of rotation, a block B weighing $p_1 = 16.1$ lbs. is attached to the free end of a thin wire which is wound around the rim. When B is released, it falls a distance $h = 6$ ft. in $T_1 = 16$ sec. To eliminate the effects of bearing friction, a second block weighing $p_2 = 8.05$ lbs. is used. This block falls 6 ft. in $T_2 = 25$ sec. Assume that the moment of friction was constant and the same in each experiment. Find the moment of inertia I. *Ans.* 9600 in. lb. sec.^{2}



659. Water enters a fixed vertical channel with a velocity $v_0 = 6$ ft./sec. at an angle $\alpha_0 = 90°$ to the horizontal. The area of the channel entrance is 0.2 sq. ft. The exit angle is $\alpha_1 = 30°$ to the horizontal and the exit velocity $v_1 = 12$ ft./sec. Find the horizontal force exerted by the water against the wall of the channel. *Ans.* 24.1 lbs.

660. An airplane weighing p lbs. lifts itself by driving downward a column of air having a cross section of a sq. ft. The air weighs q lbs. per cu. ft. With what velocity should this column move downward to hold up the airplane? What engine power is needed to do this?

$$\textit{Ans.}\quad v = \sqrt{\frac{pg}{qa}}\ \text{ft./sec.};\ P = \frac{p}{1100}\sqrt{\frac{pg}{qa}}\ \text{H.P.}$$

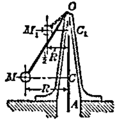


661. A rope passes over a pulley of negligible weight. Two men A and B of equal weight hang on to the free ends of the rope. A climbs up his rope with a velocity a. What happens to B if he continues to hold on to his end of the rope? *Ans.* $V_B = \frac{a}{2}$ upward.

662. A machine for studying bearing friction consists of a flywheel mounted on a journal shaft. The flywheel has a moment of inertia I. It is brought up to an angular velocity ω_0 and then permitted to coast down. It stops at the end of T seconds.

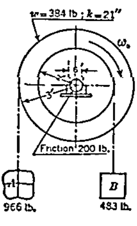
Find the frictional moment, assuming that it remains constant throughout the motion. *Ans.* $M = \dfrac{I\omega_0}{T}$.

663. A round horizontal turn-table rotates without friction around a vertical axis through its center. A man whose weight is p walks around a circumferential path of radius r. The weight P of the turn-table is uniformly distributed over the disc, the radius of which is R. At the start, both the man and the turn-table were at rest. The man walks around the turn-table with a relative velocity u. Find the angular velocity of the plate caused by the man's motion. *Ans.* $\omega = \dfrac{2pru}{PR^2 + 2pr^2}$.



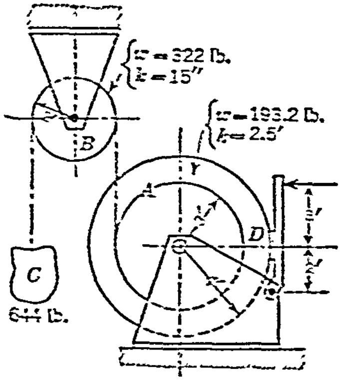
664. A weight M is attached to the end of an inelastic string MOA. A part OA of the string passes through a vertical pipe. The weight rotates about the axis of the pipe in a circular path of radius $CM = R$ at a speed of 120 r.p.m. The string is drawn slowly into the pipe until the rotating length is shortened to OM_1. The ball is then rotating at a distance $M_1C_1 = R/2$ from the axis of the pipe. Find the speed of rotation in the new position and the increase of the kinetic energy of the ball.

Ans. $\omega_1 = 4\omega_0 = 480$ r.p.m. The kinetic energy is increased fourfold.



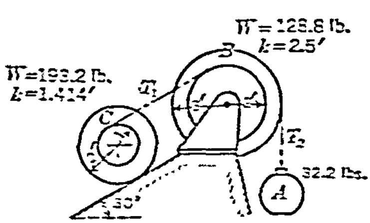
665. This drum weighs 384 lbs. and has a radius of gyration of 21 inches. The radius of the axle is 3 inches and the friction force upon it amounts to 200 lbs. (*a*) At a given instant, the drum is rotating clockwise with a speed of 60 revolutions per minute. Use the principle of Impulse and Momentum to determine the time elapsed while the speed of the drum is changing from 60 r.p.m. to zero. (*b*) What is the tension in each cord before the system stops?

Ans. (*a*) $t = 1.16$ sec.; (*b*) $T_A = 480$ lbs.; $T_B = 645$ lbs.



666. At a given instant, the downward velocity of body C is 10 ft./sec. Take the coefficient of friction at D as $f = 3/5$, and assume a 60-lb. axle friction force with an axle whose diameter is 4 in. Use the principle of Impulse and Momentum to determine the force P which must be applied to the brake to bring C to rest in ½ second.

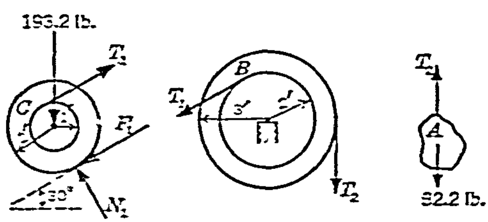
Ans. $P = 603$ lbs.



667. At a given instant, a body A has a velocity downward of 12 ft. per second. Cord BC is parallel to the 30° plane, and C rolls without slipping. Using the principle of Impulse and Momentum, find the time elapsed while the velocity of A changes from 12 to 4 ft./sec. downward.

Solution:

The elapsed time is computed, by use of the principle of Impulse and Momentum for body A (§ 128a), body B (§ 126a), and body C (§ 127).



(a) (b) (c)

If the velocity of A is v, the angular velocity of B is $v/3$, the angular velocity of C is $2v/9$, and the velocity of the center of C is $4v/9$.

$$\text{For } A: \quad (32.2 - T_2)\Delta t = \frac{32.2}{32.2}(4 - 12) = -8,$$

$$\text{For } B: \quad (3T_2 - 2T_1)\Delta t = \frac{128.8}{32.2}(2.5)^2\left(\frac{4}{3} - \frac{12}{3}\right) = -66.7,$$

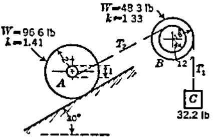
$$\text{For } C: \quad (T_1 - F_1 - 96.6)\Delta t = \frac{193.2}{32.2}\left(\frac{16}{9} - \frac{48}{9}\right) = -21.3,$$

$$\text{For } C: \quad (T_1 + 2F_1)\Delta t = \frac{193.2}{32.2}(1.414)^2\left(\frac{8}{9} - \frac{24}{9}\right) = -21.3.$$

Rearranging the above equations we have

$$\begin{aligned}(T_1 - 2F_1)\Delta t &= -\ 21.3,\\ (2T_1 + 2F_1 - 193.2)\Delta t &= -\ 42.6,\\ (4.5T_2 - 3T_1)\Delta t &= -100.0,\\ (145 - 4.5T_2)\Delta t &= -\ 36.0\\ \hline -48.2\Delta t &= -199.9,\end{aligned}$$

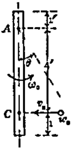
Adding we find $\qquad \Delta t = 4.15$ *seconds*



668 At a given instant, a body C has a velocity of 5 ft per second downward Using the principle of Impulse and Momentum, find the velocity of C two seconds later

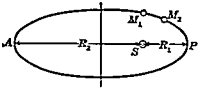
Ans $\quad v = 1.27$ ft per sec

669 A uniform thin rod 6 ft in length weighs 16.1 lbs The rod rotates in a vertical plane about a fixed axis at a point A



The rod has a counterclockwise angular velocity of 1 r p m At the instant when the rod is in a vertical position, a bullet weighing 0.01 lb is shot horizontally into the rod, towards the left, with a velocity of 2000 ft per sec The bullet enters the rod at a point C, one ft from the lower end and remains embedded in it (*a*) Calculate the subsequent angular velocity of the body (*b*) Determine whether the rod will make a full revolution, and if not, calculate the maximum angle through which it will swing

Ans $\quad$ (*a*) 0.6 rad per sec, (*b*) 11° 25′



670 Two meteorites M_1 and M_2 revolve on the same elliptical orbit, which has the sun S as one of its foci The distance between the meteorites is small and the arc M_1M_2 can be considered a straight line M_1M_2 has the length a when its middle is at the perihelion P The areas swept out by the radius vectors are equal Find the distance between M_1 and M_2 when the midpoint between them is in the aphelion A Let $SP = R_1$ and $SA = R_2$

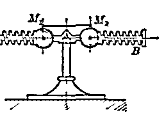
Ans $\quad M_1M_2 = a\dfrac{R_1}{R_2}$

earth is $\Delta\omega = \frac{1}{86\,400} \times \omega_{earth} = \frac{2\pi}{(86\,400)^2}$ rad/sec, the weight and radius of the earth being P and R the change in the angular momentum of the train is puR/g (§ 124), and the change in the angular momentum of the earth is (§ 127a) $I\Delta\omega = (2/5)(P/g)R^2\Delta\omega$ I is the moment of inertia of the earth about its axis of rotation The two changes in angular momentum are equal and opposite to each other $puR/g = (2/5)(PR^2/g)\Delta\omega$, $u = (2/5)(P/p)R\Delta\omega$ $= 3\ 7 \times 10^{12}$ mi/sec

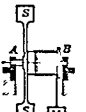
676 How much would the length T of the day change if cosmic dust falling on the earth were to cover the globe with a thin uniform layer weighing $m = 297 \times 10^{14}$ lbs Data on the earth's size and weight are given in the preceding problem

Ans $\Delta T = 0\ 00039$ sec

677 A horizontal rod AB of length $2L = 75$ in and weighing $Q = 5$ lbs is supported in the middle by a pivot Two balls M_1 and M_2, each weighing $P = 12\ 5$ lbs, can slide on the rod They are placed symmetrically with respect to the pivot and the distance between them is $2l_1 = 30$ in Two identical springs are fixed at the ends of the rod and at the inner ends they are attached to the two balls The balls are held in position by means of a string tied between them The rod AB is set spinning in the horizontal plane at a speed of $n_1 = 64$ r p m The string holding the balls is burnt and after several oscillations the balls come to rest in their new position $2l_2 = 45$ in apart Neglecting the effects of the spring masses, find the speed n_2 with which the rod rotates with the balls in their new position *Ans* $n_2 = 34$ r p m

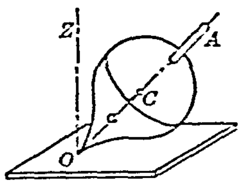


678 A drum of radius r is rotated around its axis AB by means of a weight M suspended on a rope which is wound around the drum In order to reach a constant velocity in a short time, the drum is equipped with n identical fan blades S One blade offers a resistance to motion equivalent to a force acting at a distance R from AB and proportional to the square of the angular velocity of the drum The coefficient of proportionality is l M weighs q lbs, the moment of inertia of the rotating parts is I The weight of the rope can be neglected Find the angular velocity of the drum



as a function of time, assuming that it starts from rest. Show that the velocity approaches a constant value as t increases.

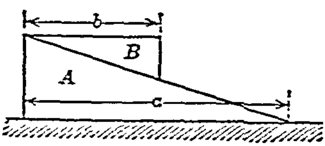
$$Ans. \quad \omega = \sqrt{\frac{qr}{nkR}}\,\frac{e^{at}-1}{e^{at}+1}, \quad \text{where} \quad a = \frac{2\sqrt{qrnkR}}{I+(q/g)r^2}.$$



679. A top spins about its axis OA in a clockwise direction at a high angular velocity ω. The axis OA is inclined to the vertical and the point of the top O stays in one position. The weight of the top is Q and its center of gravity is on the axis at a distance $OC = a$ from the point O. The moment of inertia around the axis is I. At a high value of angular velocity ω, the moment of momentum of the top can be assumed to be equal to $I\omega$ with its vector directed along OA. Describe the motion of the axis OA.

Ans. The top rotates around the OZ axis with an angular velocity $\omega_1 = Qa/(I\omega)$, clockwise.

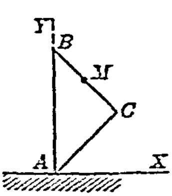
27. Motion of the Center of Gravity.



680. A prism B rests on a prism A which lies on a horizontal table, as shown in the sketch. The cross-sections of the prisms are similar right triangles. The prism A weighs three times as much as the prism B. The prisms and the table are ideally smooth. B slides down A until it touches the table. Find the distance through which the prism A moves during this action.

Solution:

Only vertical forces act on the system; the center of gravity of A and B remains therefore on the same vertical line (§ 132). While B slides to the right, A moves to the left. The horizontal displacement of B being x_B and that of A being x_A, we find $x_B + x_A = a - b$. Since the center of gravity does not move horizontally, we have $x_B = 3x_A$. Therefore $x_A = (a - b)/4$.



681. A thick plate ABC, shaped like an isosceles right triangle with an hypothenuse $AB = 12$ in., is placed with the corner A on a smooth horizontal plane. AB is vertical. The plate is released and falls in the xy plane, under the action of gravity. Find the path of the point M which is the middle of the side BC. *Ans.* $9(x-2)^2 + y^2 = 90$, an arc of an ellipse.

682 A man sits in the stern of a boat floating without motion on a lake He rises and walks to the stem The length of the boat is l, its weight is m_1 The weight of the man is m_2 Neglecting the resistance of the water, find the distance s which the boat moves while the man walks from the stern to the stem Find the ratio between the absolute velocity v of the boat and the relative velocity u of the man during the motion

Ans $s = l \times \frac{m_2}{m_1 + m_2}$, $\frac{v}{u} = \frac{m_2}{m_1 + m_2}$.

683 A man weighing p lbs jumps with an initial velocity v_0, directed at an angle α to the horizontal He holds in his hands a weight q which he throws horizontally backward with a relative velocity u at the instant he reaches his highest altitude Find the velocity v of the man just after he has thrown the weight p, and find the length s of his jump

Ans $v = v_0 \cos\alpha + uq/p$, $s = \frac{v_0 \sin\alpha}{g}\left(2v_0 \cos\alpha + \frac{uq}{p}\right)$

684 A steamer weighing 400 000 lbs moves with an average speed of 30 ft/sec The thrust of the paddle wheels always equals the water resistance The piston of the steamer's horizontal engine weighs 200 lbs, has a stroke of 3 ft, and makes 240 strokes per minute The piston has simple harmonic motion Find the velocity v of the steamer as a function of time

Ans $v = (30 + 0\ 0094 \cos 4\pi t)$ ft per sec

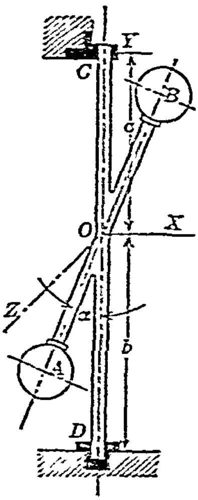
685 A train weighing 400 000 lbs rolls on to a ferry boat at a speed of 12 mi/hr The brakes are applied and the train comes to rest after traveling 75 ft Find the tension T in each of the two cables which moor the ferry boat to the dock Neglect the effects of vertical displacements *Ans* $T = 12{,}800$ lbs

686 A boat weighing p_1 moves with a velocity v_1 A weight p is thrown from the stern in the direction opposite to the motion of the boat with a relative horizontal velocity u At the instant the weight is thrown, the oarsmen stop rowing The water resistance, proportional to the square of the boat's velocity, is kv^2 In what interval of time t_1 will the boat be back to its original velocity v_1?

Ans $t_1 = \frac{pp_1u}{kgv_1(pu + p_1v_1)}$ sec.

28. Bearing Reactions.

687. A flywheel 6 ft. in diameter and weighing 6000 lbs. has its center of gravity 0.040 in. from the axis of rotation. Find the bearing reactions when the flywheel is rotating at a speed of 1200 r.p.m.



688. A rod AB of length $2l$ carrying on its ends two loads A and B, each weighing p lbs., rotates around a vertical axis OY with a uniform angular velocity ω. OZ passes through the middle O of the rod AB. $OC = a$ and $OD = b$. The angle between OY and OB is α. Find the reactions at the bearing C and the pivot D when the rod is in the plane XOY. Neglect the effects of the rod weight and the load dimensions.

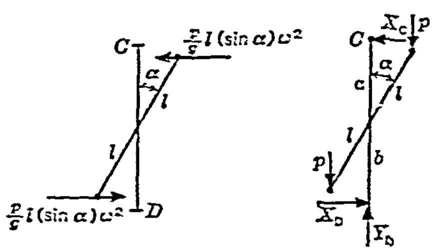
Solution:

The bearing reactions can be determined by considering the effective force system as shown in the free body diagram in (a) (§§ 133, 134*a*). The effective force system in (a) must be equivalent to the force system in (b):

$$M_{D-z} = (p/g)l \sin \alpha\omega^2 \cdot 2l \cos \alpha = X_C\,(a + b),$$

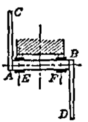
$$X_C = \frac{pl^2\omega^2 \sin 2\alpha}{g(a + b)}, \qquad X_D = X_C, \qquad Y_D = 2p.$$

This problem can also be solved by determining the resultant effective force system, or by considering the inertia force system.



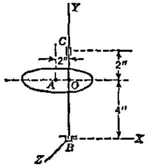
(a) (b)

689. The sprocket axle of a bicycle carries on its ends two rigidly fixed identical cranks AC and BD of length l and weight Q



extending in opposite directions. The mass of each crank may be considered as uniformly distributed along its length. The length of the axle is $2a$ and its weight is P. It rotates with a constant angular velocity in the bearings E and F. The bearings are located symmetrically and the distance between them is $2b$. Find the reactions N_E and N_F at the bearings at the instant the crank AC is directed vertically upward.

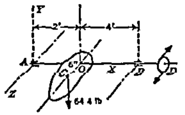
Ans. $E_y = Q + \dfrac{P}{2} - \dfrac{Qal}{2bg}\omega^2, \qquad F_y = Q + \dfrac{P}{2} + \dfrac{Qal}{2bg}\omega^2.$



690. This flywheel rotates in a horizontal plane about the vertical axis BC, the mass center being 2 inches away from the axis. The axle is supported in bearings at B and C. The flywheel weighs 644 lbs. Find the bearing reactions for the position shown, when $\alpha = 8$ rad./sec.^{2} (clockwise when looking down), and $\omega = 180$ r.p.m.

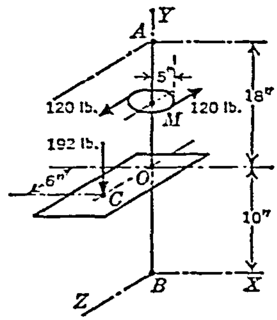
Ans. $C_x = 1003$ lbs.; $C_z = -18$ lbs.;
$B_x = 179$ lbs.; $B_y = 644$ lbs.;
$B_z = -9$ lbs.

691. A body weighing 64.4 lbs. rotates about a horizontal axis AB. The perpendicular distance from the center of gravity C to the axis is 6 inches. The figure represents the plane of symmetry of the body with mass center at C. The axis rests in bearings at A and B, and a couple is applied to the axis at D of such magnitude that A_y and B_y, the vertical components of the axle reactions, equal zero when CO is horizontal. What is the angular acceleration of the body when CO is horizontal? If the angular velocity is 6 rad./sec. at this instant, what are the magnitudes and senses of A_z and B_z?



Ans. $\alpha = 64.4$ rad. per sec.^{2}; $A_z = -24$ lbs.; $B_z = -12$ lbs.

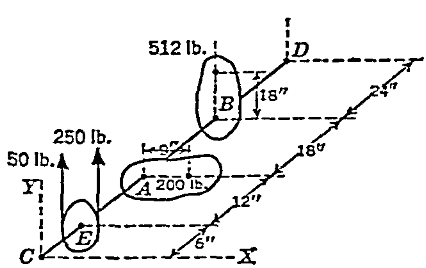
692. OC is the horizontal plane of symmetry of a body which is fastened to and rotates about the vertical axis AB. The center of gravity C of the body is 6″ from the axis of rotation. The body weighs 192 lbs. and its moment of inertia with reference to the axis of rotation is 10 lb. ft. sec.². The body is



turned by a couple at M. Neglect the mass of M. Determine the angular acceleration of the rotating body. One half second after starting from the position of rest, the body occupies the position shown. What is the angular velocity at this instant? Determine the x and z components of the reactions at A and B at this instant.

Ans. $\omega = 5$ rad. per sec.; $A_x = 11$ lbs.; $B_x = 19$ lbs.; $A_z = -68$ lbs.; $B_z = -7$ lbs.

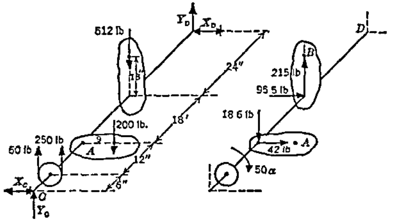
693. A and B are the planes of symmetry of two bodies which are perpendicular to the axis of rotation. The system consisting of the bodies A and B rotates with the axis CD. The pulley at E has a radius of 1 foot. The total moment of inertia of the



system is 50 lb. ft. sec.² Neglecting the masses of the shaft and pulley E, find the x and y components of the bearing reactions at C and D, when the body is in the position shown and is rotating with an angular velocity of 3 radians per second.

Solution:

The inertia forces for the rotating system shown in (b), when added to the actual forces acting on the system shown in (a), produce a condition of equilibrium (§§ 91a, 134a).



(a) (b)

The torque of all forces shown in (a) plus the torque shown in (b) equals zero:

$$(250 - 50) \times 1 - 200 \times \frac{3}{4} - 50\alpha = 0, \qquad \alpha = 1 \text{ rad./sec.}^2.$$

For body A: $Mf\omega^2 = \dfrac{200}{32\,2} \times \dfrac{3}{4} \times 3^2 = 42$ lbs.,

$$Mf\alpha = \frac{200}{32\,2} \times \frac{3}{4} \times 1 = 4.7 \text{ lbs.},$$

For body B: $Mf\omega^2 = \dfrac{512}{32\,2} \times 1.5 \times 3^2 = 216$ lbs.,

$$Mf\alpha = \frac{512}{32.2} \times 1.5 \times 1 = 23.8 \text{ lbs.}$$

Additional equilibrium equations yield the reaction forces at C and D:

$$\Sigma M_{C-x} = 300 \times 6 - 200 \times 18 - 512 \times 36 + 60\,Y_D - 4.7 \times 18 + 215 \times 36 = 0;$$

$$Y_D = 210 \text{ lbs.}$$

$$\Sigma F_y = Y_C + 210 + 300 - 200 - 512 - 4.7 + 215 = 0;$$

$$Y_C = -8.$$

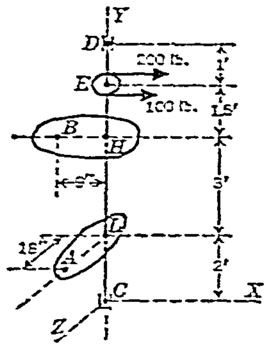
$$\Sigma M_{C-y} = 60\,X_D + 42 \times 18 + 23.8 \times 36 = 0;$$

$$X_D = -26.9 \text{ lbs.}$$

$$\Sigma F_x = X_C - 26.9 + 42 + 23.8 = 0;$$

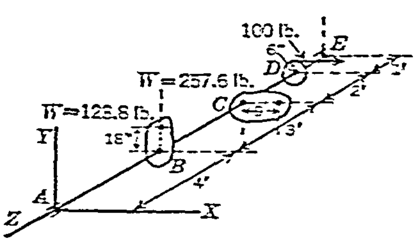
$$X_C = -38.9 \text{ lbs.}$$

694. A and B are the horizontal planes of symmetry of two bodies which are fastened to and rotate with the vertical axis CD. A weighs 128.8 lbs. and its mass center is 18 in. from the axis of rotation. B weighs 322 lbs. and its mass center is 9 in. from the axis. The moment of inertia for the rotating system



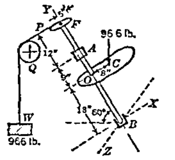
with respect to the axis of rotation CD is 12.5 lbs. ft. sec.^{2}. A torque is exerted by unequal forces on the 12-in. diameter pulley at E. If, for the position shown, the angular velocity is 3 radians per second, find the bearing reactions at C and D. (BH is in the xy plane and AL in the yz plane.)

Ans. $D_x = -190.0$ lbs.; $D_z = -60.0$ lbs.; $C_x = -66.5$ lbs.; $C_z = -23.8$ lbs.



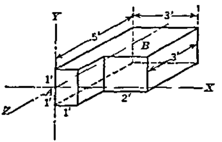
695. B and C are the vertical planes of symmetry of two bodies which are fastened to and rotate about the horizontal axis AE. B weighs 128.8 lbs. and its mass center is 18 in. vertically above the axis AE. C weighs 257.6 lbs. and its mass center is 9 in. horizontally to the right of the axis AE. The moment of inertia about the axis of rotation is 265 lb. in. sec.^{2}. The system is acted upon by the 100-lb. pull toward the right on the 12-in. pulley at D. Neglecting the masses of the shaft AE and the pulley D, find the x and y components of the bearing reactions at A and E, when the body is in the position shown and is rotating with an angular velocity of 5 radians per second.

Ans. $A_x = -15.4$ lbs.; $A_y = +44.8$ lbs.; $E_x = -168.6$ lbs.; $E_y = +125.6$ lbs.



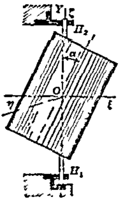
696. OC represents the plane of symmetry of a body fastened to the inclined axle AB, which makes an angle of 60° with the horizontal and lies in the xy plane. A body W hangs on a cord which passes over the pulley Q and is wound around the rim of the pulley F, which is rigidly attached to the axle. The pull in P causes OC and its axle to rotate. If the moment of inertia of OC with respect to the axis of rotation is 60 lb. in. sec.^{2} and the speed of rotation when in the position shown is 30 r.p.m., what is the angular acceleration of OC? What is the pull in cord P? What are the x and z components of the axle reaction at A and the x, y, and z components at B?

Ans. $A_x = 678$ lbs.; $A_y = 0$; $A_z = -52.5$ lbs.;
$B_x = -176$ lbs.; $B_y = 83.7$ lbs.; $B_z = -26.3$ lbs.;
$P = 473$ lbs.



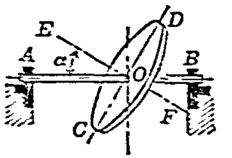
697. This homogeneous block is rotating clockwise about the horizontal axis AB. The axle is supported in bearings at A and B. The block weighs 644 lbs. Find the components of the bearing reactions for the position shown when $\alpha = -10$ rad./sec.^{2} (clockwise) and the speed of rotation is 180 r.p.m.

Ans. $A_x = -3130$ lbs.; $A_y = 163$ lbs.; $B_x = -6250$ lbs.;
$B_y = 217$ lbs.



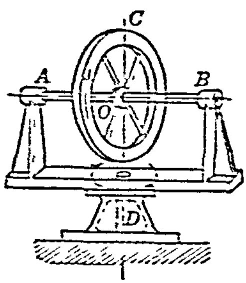
698. A solid cylinder of length $2l$ and radius r and of weight P rotates at a constant angular velocity ω around a vertical axis OY which passes through the center of gravity of the cylinder. The angle between the axis of the cylinder and OY is α. The distance between the bearings $H_1H_2 = h$. Find the lateral forces N_1 and N_2 on the bearings H_1 and H_2.

Ans. $N_1 = N_2 = \dfrac{P\omega^2 \sin 2\alpha}{2gh}(l^2/3 - r^2/4)$.



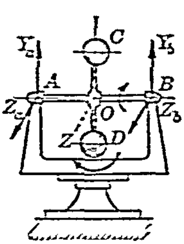
699. A turbine wheel CD is pressed onto an axle AB. Due to improper boring an angle $AOE = \alpha = 0.02$ radian is formed between the true axis EF of the wheel and the axle AB. The wheel weighs 7.08 lbs.; its radius is 8 in. $AO = 20$ in.; $BO = 12$ in. AB is assumed to be absolutely rigid. Find the forces due to this misalignment acting on the bearings A and B when the wheel is rotating at a speed of 30,000 r.p.m.

Ans. $F_A = -F_B = 1810$ lbs.



700. A wheel of radius a, weighing $2p$, rotates around its horizontal axis AB with a constant angular velocity ω_1. The axis AB in turn rotates around a vertical axis CD through the center of the wheel O with a constant velocity ω_2. $AO = OB = h$. The directions of rotation are indicated by arrows. Find the forces N_A and N_B on the bearings A and B.

$$\textit{Ans.}\quad N_A = p\left(1 + \frac{a^2\omega_1\omega_2}{2gh}\right);\ N_B = p\left(1 - \frac{a^2\omega_1\omega_2}{2gh}\right).$$



701. The device described in Problem 700 carries on its horizontal axis AB a rod CD with a ball on each end instead of the wheel. The weight of each ball is Q. $OC = OD = a$. The axis AB rotates in a horizontal plane with an angular velocity ω_1. CD rotates around AB with an angular velocity ω_2. The directions are indicated by arrows. Find the horizontal and vertical components of the bearing reactions. Neglect the weight of the rods and consider the mass of each ball concentrated at its center.

$$\textit{Ans.}\quad Y_a = Q + \frac{Qa^2}{hg}\,\omega_1\omega_2[1 + \cos 2\omega_2 t];$$

$$Y_b = Q - \frac{Qa^2}{hg}\,\omega_1\omega_2[1 + \cos 2\omega_2 t];$$

$$Z_b = -Z_a = \frac{Qa^2}{hg}\,\omega_1\omega_2 \sin 2\omega_2 t.$$

29 Vibration and Oscillation

702 A weight Q 8000 lbs, is attached to the lower end of a steel cable. The elastic properties of the cable are such that a pull of 8000 lbs stretches it 0.2 in. Find the maximum load on the cable if the weight Q is lifted just enough to make the tension in the cable zero, and is then dropped.

Solution

(1) The cable tension in terms of the cable stretch x is given by $T = 40\,000x$ lbs where x is in inches. The total work done by the weight and the cable tension acting on Q as it moves from the initial position ($v_1 = 0$) to the point of maximum cable stretch ($v_2 = 0$) is equal to zero (§ 119a).

$$8000x - \frac{40\,000x^2}{2} = 0 \qquad x = 0.4 \text{ in}$$

The maximum cable tension $=$ *16 000 lbs*

(2) The equation of motion of weight Q is

$$\frac{8000}{386}\,\frac{d^2x}{dt^2} = 8000 - 40\,000x$$

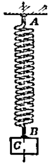
Integrating this equation (§§ 136, 136a) with

$$\frac{dx}{dt} = 0 \qquad \text{and} \qquad x = 0 \qquad \text{when} \qquad t = 0,$$

we find that the position of Q is $x = \frac{1}{5}(1 - \cos\sqrt{1930}\,t)$

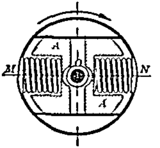
The maximum value of x is $\frac{1}{5} \times 2 = 0.4$ in

The maximum cable tension $=$ *16 000 lbs*



703 A spring AB fixed at one end requires a force of 20 grams to stretch it 1 cm. A weight C of 100 grams is attached to the free end of the spring without stretching it, and it is then dropped. Find the amplitude and *frequency of the ensuing motion*. Neglect the effect of the weight of the spring.

Ans 5 cm, 2.23 cycles per sec



704 A governor consists of two 60-lb weights A attached to the ends of two springs. The weights are guided along the line MN and the springs are fixed at M and N. The centers of gravity of the two weights are located at the inner ends of the springs which are 2 in from O when the springs are not compressed. A force

of 100 lbs. will compress each spring 1 in. Find the natural frequency of oscillation of the weights A when the vertical governor spindle O is rotating uniformly at a speed of 120 r.p.m.

Ans. 3.51 cycles per sec.

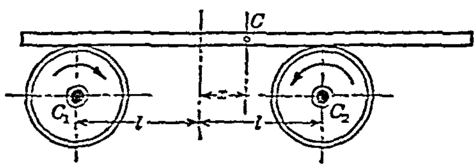
705. The springs of a car each carry a load of P lbs. They all deflect 2 in. under this load. The compression of each spring is proportional to the load on it. Find the frequency f of the vertical oscillation of the car. *Ans.* $f = 2.21$ cycles per sec.

706. A vertical rod, fixed at its lower end B, carries a weight $Q = 6$ lbs. attached to its upper end A. If the loaded end is displaced from its position of equilibrium and released, it will oscillate. It is found by means of a spring balance that a horizontal force of 0.6 lb. applied at A will deflect that point 1 in. Find the natural frequency f of horizontal oscillation of the weight Q when the motion is so small that the weight can be considered as moving in a straight line. *Ans.* 1 cycle per second.

707. An elastic thread fixed at one end carries a load weighing p oz. at its free end. If the loaded thread is stretched and released, the load will oscillate. The elongation of the thread is proportional to the force producing it, and a force of q oz. will stretch it 1 in. The unstretched length of the thread is l. The thread is stretched to a length x_0 and released. It starts to oscillate with an initial velocity of zero. Define the length x of the thread as a function of time. Find the range of magnitude of the initial length x_0 for which the thread will be in tension throughout the motion.

Ans. $x = l + p/q + (x_0 - l - p/q) \cos \sqrt{\frac{qg}{p}}\, t; \; l < x_0 < l + \frac{2p}{q}.$

708. Two pulleys of equal radii rotate in opposite directions. Their centers C_1 and C_2 are on a horizontal line C_1C_2. $C_1C_2 = 2l = 20$ in. A rod laid on the pulleys is acted upon by frictional



forces at the points of contact. These forces are proportional to the loads on the contact points. The coefficient of friction is f.

714. A pendulum consists of a round rod AB of weight p, length l and radius r, and a coaxial weight CD of the same material of length l_1 and radius r_1. The axis of rotation EF is at a distance a from the upper end A of the rod. The lower base D of the weight is at a distance b from the lower end B of the rod. Assume $p = 3$ lbs., $l = 24$ in., $r = 0.4$ in., $l_1 = 4.8$ in.; $r_1 = 0.8$ in., $a = b = 2$ in. Find the frequency of small oscillations of this physical pendulum.

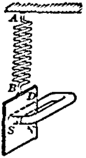
Ans. 0.78 cycles/sec.

715. A pendulum consists of a rod AB and a ball C of radius r and weight p. The center of the ball is on the center line of AB. Neglecting the weight of the rod, determine the location of the suspended point O to obtain small oscillations having a desired period T.

Ans. $OC = \frac{1}{8\pi^2}(gT^2 + \sqrt{g^2T^4 - 25.6\pi^4 r^2})$

716. A pendulum consists of two balls mounted on the ends of a rod. The length of the rod is l. The upper ball weighs p_1 and the lower weighs p_2. At what distance x from the lower ball should the point of suspension be put to get the maximum frequency of oscillation? Neglect the weight of the rod and the dimensions of the balls.

Ans. $x = l\sqrt{p_1}\left(\frac{\sqrt{p_1} + \sqrt{p_2}}{p_1 + p_2}\right)$



717. A spring AB fixed at its upper end A has a plate D, weighing 100 grams, suspended on the free end. The plate hangs between the poles of a magnet. Any motion of the plate is resisted by the action of eddy currents in the plate. The resistance has the value $k\Phi^2 v$ dynes, where $k = 0.0001$, Φ is the magnetic flux between the poles of the magnet, and v is the velocity of the plate in cm. per sec. The elastic properties of the spring are such that a force of 20 grams will elongate it 1 cm. At time $t = 0$, the plate is raised

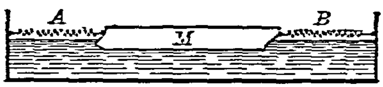
until the tension in the spring is zero, and is then dropped. Give the equation of motion of the plate when $\Phi = 1000\sqrt{5}$ maxwells.

Ans. $x = [5 - e^{-2.5t}(5\cos 13.8t + 0.91 \sin 13.8t)]$ cm.

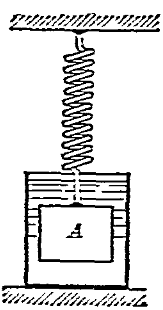
718. Find the motion of the plate D in the previous problem when the magnetic flux is $\Phi = 10{,}000$ maxwells.

Ans. $x = [5 - (5/48)(49e^{-2t} - e^{-98t})]$ cm.

719. To determine the resistance of water to the motion of ships at low velocities, a small model of a boat M is placed in a tank and two identical springs A and B are attached to the stem and stern and to the ends of the tank. Any displacement of the model from its position of equilibrium is proportional to the force acting on it. The model is displaced from its position of equilibrium and it is noted that the resulting oscillations grow smaller in geometric proportion, the ratio between successive amplitudes being 0.9. This shows that the frictional resistance is proportional to the velocity. The frequency of oscillation is 322/336 cycles per second. Find the frictional resistance in ounces per ounce of the model weight at a velocity of 1 in./sec.



Ans. 0.00104 oz. per oz.



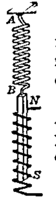
720. Coulomb used the following method to determine the viscosity of liquids. A thin plate weighing P lbs. was suspended on a spring and made to oscillate first in air and then in the liquid being tested. The periods of oscillation T_1 and T_2 in both cases were observed. The frictional resistance in air was negligible. In the liquid it was equal to $2Sfv$, where $2S$ was the total surface of the plate, f was the coefficient of viscosity and v was the velocity of the plate. Find f as a function of T_1 and T_2.

$$Ans.\quad f = \frac{2\pi P}{gST_1T_2}\sqrt{T_2^2 - T_1^2}.$$

721. A body A weighing 1.2 lbs. lies on a rough surface and is attached to a fixed point B by means of a horizontal spring BC. The coefficient of friction on the surface is 0.2. A force of 1.5 lbs. will stretch the spring 1 in. The body A is displaced

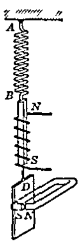
1 2 inches to the right, and is released How many swings will it make? What is the length of each swing? What is the duration of each swing?

Ans 4 swings Length of the swings $l_1 = 2\ 08$ in , $l_2 = 1\ 44$ in , $l_3 = 0\ 80$ in , $l_4 = 0\ 16$ in The duration of each swing is 0 143 sec



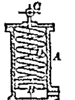
722 In Problem 703, the weight C is replaced by a magnetic rod weighing 100 grams The lower end of the rod extends into a solenoid through which alternating current is flowing The current $i = 20 \sin (2\pi t/T)$, where $T = 0\ 25$ sec The axial force exerted on the rod is $F = 16\pi i$ dynes At time $t = 0$, the rod is hanging in its position of static equilibrium and the current is switched on Find the forced oscillation of the magnet

Ans $x = -0\ 023 \sin 8\pi t$ cm



723 In the previous problem, assume the magnetic rod to weigh only 50 grams and that a plate weighing 50 grams is hung from it below the solenoid The same force acts on the rod as in Problem 722 The plate moving between the magnet poles gives rise to a braking force of $k\Phi^2 v$ dynes, where $\Phi = 1000\sqrt{5}$ maxwells, $k = 0\ 0001$ Find the forced oscillation of the plate

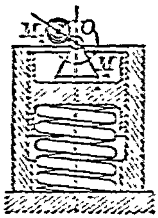
Ans $x = -0\ 022 \sin (8\pi t + 0\ 91\pi)$ cm



724 A steam indicator consists of a cylinder A, and a piston B acting against a spring D and carrying a rod BC that operates a recording pencil The moving parts of the indicator weigh 2 lbs and it takes 7 5 lbs to compress the spring 1 in The area of the piston is 0 6 sq in The steam pressure in lbs per sq in acting on the piston is $p = 60 + 45 \sin (2\pi t/T)$ T is the

duration of one revolution of the engine shaft. Find the amplitude of the pencil motion when the engine is running at 180 r.p.m.

Ans. 4.76 in.



725. An electric motor is mounted on a platform which can move vertically between guides. The platform rests on a helical spring. The motor and platform weigh 65 lbs. A force of 151.5 lbs. will compress the spring 1 in. The shaft of the motor carries an eccentric load weighing 0.4 lb. Its center of gravity is at a distance of ½ in. from the axis of rotation. The motor rotates with an angular velocity of 30 rad. per sec. Find the forced vibration of the platform.

Solution:

The vertical component of the centrifugal force of the eccentric is $(0.4/386)(1/2)30^2 \sin 30t = 0.466 \sin 30t$ lbs., if $t = 0$ is chosen when OM_1 is horizontal. The force acting on the platform is $F = (-151.5x + 0.466 \sin 30t)$ lbs. where x is the displacement of the platform from its static equilibrium position. The equation of motion of the platform is:

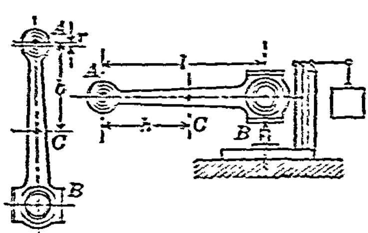
$$\frac{65}{g} \cdot \frac{d^2x}{dt^2} = F = -151.5x + 0.466 \sin 30t,$$

or

$$\frac{d^2x}{dt^2} = -900x + 2.77 \sin 30t.$$

This is the equation of a forced oscillation (§ 138), with $p^2 = 900$, and $q = 30$. Since p is equal to q, we have resonance, and the amplitude of the forced oscillation will grow indefinitely with time.

$x = 0.046t \cos 30t$ *(90° phase difference with the position of the eccentric).*



726. The following method is used to determine the moment of inertia of a connecting rod about its center of gravity. A thin pin is passed through the wrist-pin bushing and the rod is permitted to oscillate about this horizontal axis. The duration of 50 complete oscillations is 100 sec. The distance $AC = h$ between the point of suspension and the center of gravity of the rod is found by suspending the point A from a crane and placing the crank end of the rod on the platform of a scale, as shown in the sketch. The rod is held horizontal and the scale indicates a

weight of $p = 100$ lbs The distance AB between pin centers is $l = 3$ ft, the weight of the rod is $Q = 160$ lbs The radius of the wrist-pin bushing is $r = 1.5$ inches Find the moment of inertia I_C of the connecting rod about an axis passing through the center of gravity in a direction parallel to the wrist-pin bushing

Ans $I_C = 150$ in lb sec²

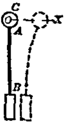
727. The suspension point of a mathematical pendulum of length l moves vertically with a uniform acceleration Find the *period of small oscillations of the pendulum under two conditions* (1) When the acceleration of the suspension point is upward and has any value p (2) When the acceleration of the suspension point is downward with any value $p < g$

$$Ans \quad (1)\ T = 2\pi\sqrt{\frac{l}{g+p}},\quad (2)\ T = 2\pi\sqrt{\frac{l}{g-p}}$$

728 A mathematical pendulum of length $OM = l$ is deflected through an angle α from its position of equilibrium OA at time $t = 0$ Its velocity at that time is zero At this instant the suspension point O, which can move vertically, has a velocity of zero and is starting to fall with a constant acceleration $p > g$ Under these conditions, find the length s of the arc through which M will swing around O

Ans $s = 2l(\pi - \alpha)$

729. *A pendulum performs small oscillations in a car moving* on a horizontal straight track The middle position of the pendulum is inclined to the vertical at an angle of 6° Find the *acceleration a of the car* Find the difference between the period T of the pendulum under these conditions and its period T_1 when the car is standing still *Ans* $a = 3.38$ ft/sec², $T : T_1 = 0.997$



730 A 6-lb weight C is attached to the upper end A of a flexible vertical rod AB A horizontal force of 0.6 lb will deflect the upper end of the rod 1 in The support at the lower end of B oscillates horizontally with an amplitude of 0.040 in and a frequency of $\frac{1}{1.1}$ cycles per second Find the amplitude of the forced oscillation of C

Solution:

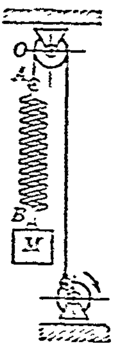
The equation of motion is:

$$\frac{6}{g}\frac{d^2x}{dt^2} = -0.6(x - x_1), \qquad \frac{d^2x}{dt^2} = -0.1gx + 0.004g\sin\frac{2\pi}{1.1}t.$$

This is the equation of forced vibration (§ 138):

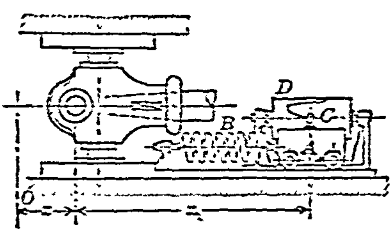
$$p^2 = 0.1g, \qquad h = 0.004g, \qquad q = \frac{2\pi}{1.1}.$$

$$Amp. = \frac{0.004g}{0.1g - \frac{4\pi^2}{(1.1)^2}} = 0.26 \text{ in.}$$



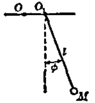
731. A body M, weighing w lbs., hangs on a spring AB, the upper end of which is moving along the vertical line OA so that its position is $x = a\cos nt$. The length of the spring unloaded is l; q lbs. stretches the spring 1 in. The initial velocity of M is zero. Assuming the spring to remain in tension throughout the oscillation, find the forced motion of M.

Ans. $x_1 = \dfrac{agq}{qg - wn^2}\cos nt.$



732. The following instrument is proposed to record the acceleration of a steam engine piston. A weight A, rolling on guided wheels mounted on the cross head, is held in place by means of a spring B. A pencil C attached to A makes a record on a chart moving at right angles to the motion of A on drum D which is also carried on the cross head. Relative motions between A and C are recorded. A weighs Q oz.; the spring characteristic is f oz. per in.; the free length of the spring is l in. A clockwork drives the drum so that the paper moves at the rate of 1 in. per sec. The motion of the cross head is $x = (a + 10\cos 20t)$ in. Find the equation of the graph traced by the pencil.

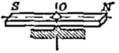
Ans. $x_1 = A\cos\left(\sqrt{\dfrac{fg}{Q}}\,y_1 + D\right) + \dfrac{4000Q}{fg - 400Q}\cos(20y_1).$



733. The suspension point of a pendulum of length l oscillates horizontally so that its distance from a fixed point O is given by the equation $OO_1 = a \sin pt$. At time $t = 0$ the pendulum is at rest. Find the motion of the pendulum, the amplitude being small.

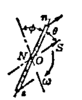
$$Ans.\quad \phi = \frac{ap^2}{(g/l) - p^2}\left(\sin pt - p\sqrt{\frac{l}{g}}\sin\sqrt{\frac{g}{l}}\,t\right).$$

734. A bar magnet $2a$ cm. long and $2b$ cm. wide weighing m grams rests on a pivot at its center of gravity. When it is displaced through a small angle from its position of equilibrium in the north and south direction, it oscillates in the field of terrestrial magnetism. The horizontal component of the magnetic field has an intensity of H gausses. The magnetic moment of the magnet, which is the product of the pole strength and the distance $2a$ between the poles, is A dyne-cm. per gauss. Find the motion of the magnet.



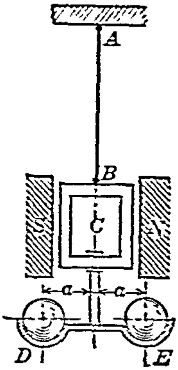
$$Ans.\quad \phi = \phi_0 \sin\left(\sqrt{\frac{3HA}{m(a^2+b^2)}}\,t + \psi\right).$$

735. A horizontal bar magnet ns rests on a pivot at its center of gravity. It is placed in a uniform magnetic field of intensity H, which rotates at a constant angular velocity ω. The magnetic moment of the magnet is A; its moment of inertia about the axis of rotation is I. The center line of the magnet is at an angle θ from the NS direction of the field and makes an angle ϕ with a fixed horizontal line in the plane SON. At time $t = 0$, the magnet is rotating at the same angular speed as the field and forms an angle θ_0 with the NS direction of the field. θ_0 is small; hence we may use $\sin\theta_0 = \theta_0$. Find the motion of the magnet. Find the length of the mathematical pendulum which oscillates in the gravitational field with the same frequency as the magnet oscillates in the rotating magnetic field.



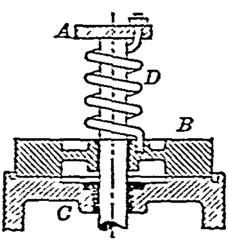
$$Ans.\quad \phi = \omega t + \theta_0 \cos\sqrt{\frac{AH}{I}}\,t;\ l = \frac{Ig}{AH}.$$

736. The coil BC of a galvanometer is suspended on a thin thread AB between the poles of a magnet. It carries two small



cups outside the magnetic field. The distance between the centers of the cups is $2a$. The coil is shorted through a resistance. Two balls of radius r and weight w are placed in the cups; the coil is turned in the magnetic field and released. The coil oscillates with a period T_1; the ratio between successive deviations from the position of equilibrium is $e^{-\delta_1}$. When the cups are empty the period of oscillation is T_2 and the ratio of deviations is $e^{-\delta_2}$. The twisting moment of the thread is $k\phi$, where k is a constant and ϕ is the angle of twist. The moment of air friction and of electric damping is $n_1\omega$ in the first case and $n_2\omega$ in the second case. ω is the angular velocity of the coil. Find the moment of inertia of the coil with respect to its axis of rotation. Find the values of the coefficients k, n_1, and n_2. Determine their numerical values when $T_1 = 11$ sec.; $\delta_1 = 0.13$; $T_2 = 4.5$ sec.; $\delta_2 = 0.30$; $a = 1.88$ cm.; $r = 0.5$ cm. and $w = 4$ grams.

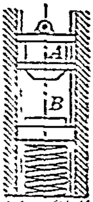
Ans. $I = 6.03$ gr. cm.^{2}; $k = 2.94$ dyne-cm./radian; $n_1 = 0.85$ dyne-cm./rad. per sec.; $n_2 = 0.80$ dyne-cm./rad. per sec.



737. A vertical shaft A drives a disc B through a spring D. The disc slides on a fixed plate C. The angular velocity of the shaft is ω. The moment of inertia of B with respect to the axis of the shaft is I. For a twist of one radian the twisting moment in the spring is n. The moment of friction f between the disc and the plate is constant. Find the relative motion between the disc and the shaft.

$$\textit{Ans.} \quad \theta - \phi = \omega\sqrt{\frac{I}{n}}\sin\left(\sqrt{\frac{n}{I}}\,t\right) - f/n.$$

30. Impact.



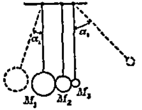
738. The hammer A of an impact machine falls from a height of 16.1 ft. and strikes an anvil B mounted on springs. The weight of the block is 20 lbs.; the weight of the anvil is 10 lbs. Find the velocity after impact of the anvil and hammer moving together.

Ans. 21.5 ft./sec.

739. The anvil of a steam hammer and the forging on it have a total weight of 500,000 lbs. The hammer weighing 24,000 lbs. falls on the forging with a velocity of 15 ft. per sec. Find the work S_1 absorbed by the forging and the work S_2 lost in vibrations of the foundation. *Ans.* $S_1 = 79{,}900$ ft.-lbs.; $S_2 = 3840$ ft.-lbs.

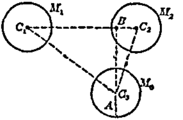
740. Find the ratio of weights m_1 and m_2 of two balls in the following cases: (1) One ball is at rest. The other ball strikes it centrally and stops while the first ball moves away. (2) The balls meet with equal and opposite velocities. After the central impact the second ball stops. The coefficient of restitution is k in both cases.

Ans. (1) $\dfrac{m_2}{m_1} = k$; (2) $\dfrac{m_2}{m_1} = 1 + 2k$.



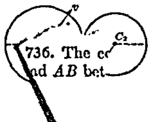
741. Three ivory balls M_1, M_2, and M_3 are suspended on thin rigid wires with their centers on the same level. The balls touch each other; the radii of the balls are in the ratio $3:2:1$. M_1 is deflected through an angle α_1 in the plane of the suspending threads and dropped. The coefficient of restitution is $k = 0.9$. Find the angle α_3 through which the ball M_3 will move. Find the values of α_1 such that the ball M_3 will return on the arc on which it swings outward.

Ans. (1) $\sin\dfrac{\alpha_3}{2} = 2.47\sin\dfrac{\alpha_1}{2}$; (2) $\alpha_1 < 48°$.



742. Three identical billiard balls M_1, M_2, and M_3 of radius R rest on a table. The balls can be considered as perfectly elastic. $C_1C_2 = a$. Find the line AB perpendicular to C_1C_2 on which to place M_3 so that when it is sent in the direction AB it will collide with M_2 and then have a central impact with M_1.

Ans. $BC_2 = 4R^2/a$.



743. Two balls move with parallel and equal velocities v in opposite directions. At the instant they collide the velocities are directed at an angle α with the line of centers C_1C_2. The mass of C_1 is twice the

mass of C_2 and the coefficient of restitution is 0.5. Find the velocities v_1 and v_2 of the balls after impact.

Solution:

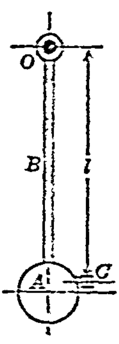
The components of velocity of C_1 and of C_2 perpendicular to line C_1C_2 are $v_n = v \sin \alpha$ (§§ 144, 145).

Consider the motion in the direction C_1C_2; if v' and v'' are the initial velocities, and u' and u'' are the final velocities, we have

$$M_1 v \cos \alpha - M_2 v \cos \alpha = M_1 u' + M_2 u'',$$
$$(u' - u'') = -0.5[v \cos \alpha - (-v \cos \alpha)],$$
$$M_2 = \tfrac{1}{2} M_1,$$
$$v \cos \alpha = 2u' + u'',$$
$$v \cos \alpha = u'' - u',$$
$$\overline{u' = 0 \qquad u'' = v \cos \alpha.}$$

Thus C_1 moves normal to line C_1C_2 with a velocity $v \sin \alpha$, and C_2 moves with a component of velocity normal to C_1C_2 equal to $v \sin \alpha$ and a component toward the right equal to $v \cos \alpha$.

The velocity of C_2 is $v_2 = v$, *at an angle* $(180° - \alpha)$ *to* C_2C_1.

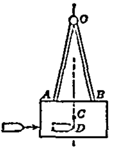


744. The pendulum of an impact testing machine consists of a steel disc of 4 in. radius and 2 in. thick suspended on the end of a round steel rod B, 0.8 in. in diameter and 36 in. long. The test piece is placed so that the direction of impact is horizontal. At what distance l from the horizontal plane containing the axis of rotation O must the test piece be placed to keep the pin O from experiencing any impact when the blow is delivered? *Ans.* $l = 39$ in.

745. Two pulleys rotate in the same plane with angular velocities $\omega_{1,0}$ and $\omega_{2,0}$. Find the angular velocities of the pulleys ω_1 and ω_2 after a belt is thrown over them. Consider the pulleys as uniform discs of equal thickness but of radii R_1 and R_2, and neglect the effects of belt slippage.

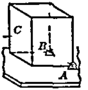
$$Ans. \quad \omega_1 = \frac{R_1^4\omega_{1,0} + R_2^4\omega_{2,0}}{R_1(R_1^3 + R_2^3)}; \; \omega_2 = \frac{R_1^4\omega_{1,0} + R_2^4\omega_{2,0}}{R_2(R_1^3 + R_2^3)}.$$

746. A ballistic pendulum used to determine shell velocities consists of a cylinder AB suspended on a horizontal axis O. The



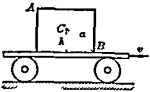
cylinder is open at one end A and is filled with sand The weight of the pendulum is M, its center of gravity C is at a distance $OC = h$ from the axis of rotation O, the radius of gyration with respect to O is k A shell striking the open end of the cylinder imbeds itself in the sand and rotates the pendulum around O through an angle α The weight of the shell is m, its distance from O is $OD = a$ Find the velocity v of the shell when it struck the pendulum, assuming that the pin O does not experience any impact, i e , $ah = k^2$

$$Ans \quad v = \frac{2(Mh + ma)}{m}\sqrt{\frac{g}{a}} \sin \alpha/2$$



747. A solid prism with a square base stands on a horizontal plane It is hinged about the edge AB on the plane The height of the prism is $3a$ and the base is a on a side, the weight is $3w$ A ball of weight w strikes the middle of the side C with a velocity v The impact is inelastic and the ball imbeds itself in the prism just at the surface at C Find the velocity v for which the prism will just tip over

$$Ans \quad v = \tfrac{1}{3}\sqrt{53ga}$$



748 A flat car carrying a load AB runs on level rails with a velocity v At B there is a cleat preventing the load from slipping forward on the car but not hindering rotation about the edge B The weight of the load is p, its center of gravity C is at a distance h above the floor of the flat car, $CB = a$, the radius of gyration of the load with respect to B is k *The flat car strikes an obstacle which stops it instantly* Find the angular velocity ω of the load around B at the instant of impact

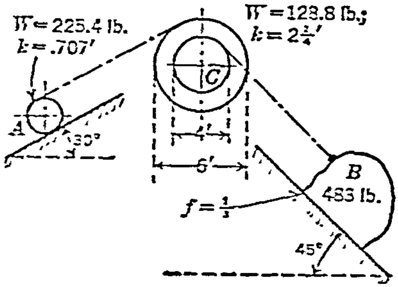
Solution.

The angular momentum of the body about B does not vary during the impact (§ 144) Before the impact the angular momentum was $(p/g)vh$ after the impact it was $I\omega$ where $I = (p/g)k^2$ is the moment of inertia of the load about B, and ω is its angular velocity just after the impact. Then

$$\frac{p}{g}vh = \frac{p}{g}k^2\omega, \qquad \text{whence} \qquad \omega = \frac{vh}{k^2}$$

749. Assume that the load in the previous problem is a uniform right rectangular prism 12 ft. long on the side which is parallel to the track and 9 ft. high. Find the velocity v with which the load will just tip over B. *Ans.* $v = 18.3$ mi./hr.

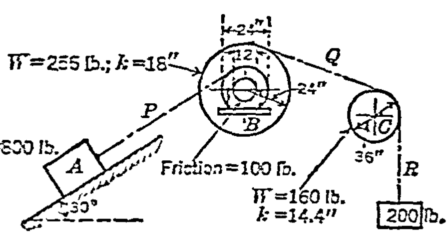
31. Review Problems.



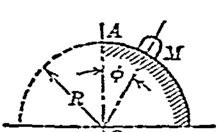
750. At a given instant the center of cylinder A of 2 ft. diameter has a downward velocity parallel to the plane of 15 ft./sec. (*a*) How far will the center of A move before stopping? (*b*) What is the tension in the cord BC during this time? Cylinder A rolls without sliding. Use: (1) force and acceleration method; (2) principle of work and kinetic energy; (3) principle of impulse and momentum.

Ans. $s = 10.5$ ft.; $T = 241$ lbs.

751. At a given instant, the body A has a velocity up the plane of 8 ft./sec. Determine how far body A will move before its velocity is reduced to 5 ft./sec. Find the tension in the

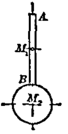


cord Q during this time. Use: (1) force and acceleration method; (2) principle of work and kinetic energy; (3) principle of impulse and momentum. *Ans.* $s = 8.37$ ft.; $T = 244$ lbs.



752. A stone M lying on the top A of a smooth hemispherical dome of radius R is given an initial velocity v_0. Where will it leave the surface of the dome? At what value of v_0 will it leave the dome as soon as it starts to move?

Ans. $\phi = \cos^{-1}(2/3 + v_0^2/3gR)$. It will leave the dome as it starts if $v_0 \geqq \sqrt{gR}$.



753 A body consists of a rod AB, 32 in long, weighing 2 lbs, and a disc of radius 8 in, weighing 4 lbs At time $t = 0$, the rod is vertical, the center of gravity of the rod M_1 has a velocity of zero and the center of gravity M_2 of the disc has a velocity of 12 ft per sec directed horizontally to the right Find the consequent motion of the body under the action of gravity alone

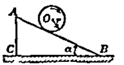
Ans The path of the center of gravity is a parabola $y = -0\ 251x^2$ The angular velocity $\omega = 6$ rad per sec

754 Two balls M_1 and M_2, weighing $p_1 = 4$ lbs and $p_2 = 2$ lbs, are connected by a rod 2 ft long At time $t = 0$, the rod is horizontal, M_2 has a velocity zero and the velocity of M_1 is $v_1 = 2\pi$ *ft per sec directed vertically upward Neglect air resistance, the weight of the rod, and the dimensions of the weights* Find the motion of the weights under the action of gravity, the distances h_1 and h_2 *of the weights from the horizontal line through* their original position at $t = 2$ sec, and the tension in the rod

Ans $y = \frac{2}{3}v_1t - g\ t^2/2$ is the motion of the center of gravity The balls move around their center of gravity with an angular velocity of $\omega = \pi$ rad/sec At $t = 2$ sec, $h_1 = h_2 = \bar{h} = -56$ ft

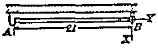
755. The coefficient of rolling friction of a ball on an inclined plane is f Find the maximum value of the angle α of inclination of the plane for which the ball will roll down without sliding

Ans $\alpha \leqq \tan^{-1}\frac{7}{2}f$



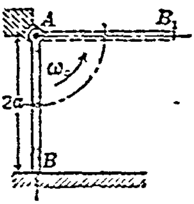
756 A triangular prism ABC of weight P is placed on a smooth horizontal plane A cylinder O of weight p rolls down the face AB without slipping Find the motion of the prism

Ans The prism moves to the left with an acceleration

$$a = \frac{gp \sin 2\alpha}{2P \cos^2 \alpha + (P + p)(1 + 2 \sin^2 \alpha)}$$

757. A beam AB of length $2l$ and of weight Q is hinged at B and held at A When released at A, it starts to fall, rotat-

ing around B. At the instant it is vertical, the end B is released. Find the path of the center of gravity and the angular velocity ω in the subsequent motion.

Ans. A parabola $\bar{y}^2 = 3l\bar{x} - 3l^2$; $\omega = \sqrt{\dfrac{3g}{2l}}$.



758. A rod AB of length $2a$ is suspended at A; the end B just clears the floor. The rod is given an initial angular velocity ω_0 and the end A is released at the instant the rod reaches a horizontal position. The subsequent motion of the rod is under the action of gravity alone. Find the initial angular velocity ω_0 for which the rod, while falling, will strike the floor in a vertical position.

Solution:

Using the principle of work and kinetic energy (§ 119a), we find that the angular velocity of the bar for the instant at which it is released is given by

$$\frac{1}{2} I_A(\omega^2 - \omega_0^2) = -Wa. \qquad I_A = \frac{4}{3}\frac{W}{g}a^2. \qquad \omega_0^2 = \omega^2 + \frac{3g}{2a}.$$

From the instant the bar is released, it continues to move with the angular velocity ω (§ 131). The center of gravity of the bar (§ 132) moves in a vertical line with an acceleration $a_y = -g$ and an initial velocity $v_0 = a\omega$, and we have

$$v_y = -gt + v_0 = -gt + a\omega,$$
$$y = \frac{1}{2}gt^2 + a\omega t + yo = -\frac{1}{2}gt^2 + a\omega t + 2a.$$

In order for the rod to strike the floor in a vertical position, the rod must turn through an angle $\pi/2$, $3\pi/2$, $5\pi/2$, $\cdots \dfrac{2k+1}{2}\pi$, $k = 0, 1, 2, 3, \cdots$,

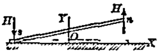
$$\omega T = \frac{2k+1}{2}\pi. \qquad T = \frac{2k+1}{2\omega}\pi.$$

Substituting this value of T in the equation for y, since $y = a$ when $t = T$, we find

$$a = -\frac{g}{2}\left[\frac{2k+1}{2\omega}\pi\right]^2 + a\omega\left(\frac{2k+1}{2\omega}\right)\pi + 2a,$$

and, since $\omega^2 = \omega_0^2 - \dfrac{3g}{2a}$,

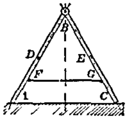
$$\omega_0^2 = \frac{g}{4a}\left[6 + \frac{(2k+1)^2\pi^2}{(2k+1)\pi + 2}\right], \quad k = 0, 1, 2, 3, \cdots.$$



759 A thin bar magnet of length $2l$ and weight P, with its poles at its two ends, lies on a horizontal plane with the poles of an electromagnet directly above and below it When the electromagnet is excited, a uniform field of intensity H acts on the bar magnet When $H > P/2$, the rod will move Find the maximum value of H for which one end of the bar magnet will stay on the plane throughout the motion Assuming that the south pole rests on the plane during the motion, find the path of the north pole when the field forces are directed upward Find the velocity of the center of gravity and the angular velocity of the bar magnet in its vertical position

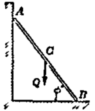
Ans (1) $H \leqq 7/12\,P$ (2) An ellipse with semi axes l and $2l$

(3) $A + \phi = 90°$, $\bar{v} = 0$, $\omega = \sqrt{\frac{6g}{Pl}}(2H - P)$



760 A step ladder ABC, hinged at B, stands on a smooth horizontal floor $AB = BC = 2l$ Each half of the ladder weighs p, the centers of gravity are in the mid points D and E The radius of gyration of each part with respect to its center of gravity is k The hinge B is at a distance h above the floor The two parts of the ladder are held together by means of a rope FG The rope breaks and the ladder collapses Neglecting the effects of hinge friction, find the velocity of the point B at the instant it hits the floor

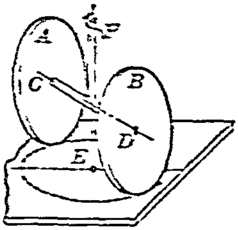
Ans $v_B = 2l\sqrt{\frac{gh}{k^2 + l^2}}$



761 A ladder AB leans against a smooth vertical wall and stands on a smooth horizontal floor It is placed with the angle $\phi_0 = 60°$ and released Find the angular velocity of the ladder at the instant $\phi = 45°$ and the angle ϕ when the force on the wall becomes zero

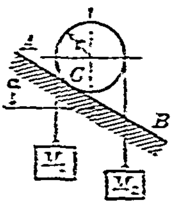
Ans $\omega = \sqrt{\frac{3g}{2l}(\sin 60° - \sin 45°)}$,

$R_1 = 0$ when $\phi = 35°$.



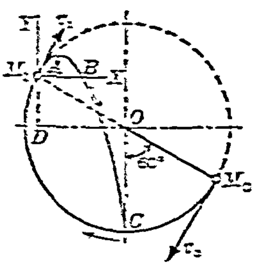
762. Two millstones A and B are mounted on a horizontal axis CD which rotates about a vertical axis EF. Each stone has a diameter of 3 ft. and weighs 400 lbs. The distance between the stones is $CD = 3$ ft. When the shaft EF rotates, the stones roll over a horizontal surface. Considering the stones as flat discs, find their kinetic energy when EF rotates at a speed of 20 r.p.m.

Ans. $E = 214$ ft.-lbs.



763. A cylinder C of radius r and weight W rolls down a plane inclined at an angle α to the horizontal. A rope thrown over the cylinder carries at its free ends two loads M_1 and M_2 of weights w_1 and w_2. The cylinder starts from rest. Find the angular velocity of the cylinder at the end of n revolutions.

Ans. $$\omega^2 = \frac{2\pi n g}{r} \cdot \frac{(W + w_1 + w_2)\sin\alpha + w_2 - w_1}{\frac{3}{2}W + w_1 + w_2 + (w_2 - w_1)\sin\alpha}.$$



764. A weight of 2 lbs. is suspended on a string 20 in. long, the other end of which is fixed at O. The pendulum is displaced 60° from its position of equilibrium to a position M_0. There it is given an initial velocity of 140 in. per sec. in a vertical plane, downward and normal to the string. Find the position M_1 of the weight where the tension in the string is zero and the velocity v_1 at this position. Find the path of the weight after this position is reached until the string is in tension again.

Ans. M_1 is located at 10.25 in. above line OD; $v_1 = 66$ in./sec. The path of the weight after it passes M_1 is a parabola; the weight moves freely under the action of gravity.

765. A helical slot with a pitch angle α is cut in the surface of a right circular cylinder which can rotate about its axis without frictional resistance. A small ball is put in the slot and rolls down along the helix. The weight of the cylinder is W; its radius is R and its moment of inertia can be taken to be $(W/g)R^2/2$. The weight of the ball is w; its distance from the axis can be

taken to be R. At the instant the ball starts to roll down, the cylinder is at rest. Find the angular velocity of the cylinder after the ball has fallen through a height h.

$$Ans. \quad \omega = \frac{2w \cos \alpha}{R} \sqrt{\frac{2gh}{(W + 2w)(W + 2w \sin^2 \alpha)}}.$$

766. From a vertical pipe standing in the center of a fountain, water jets are thrown out at different angles ϕ to the horizontal. The fountain is filled to the brim and the pipe stands 3 ft. above the water level. The water leaves the small holes in the pipe with a velocity of $\sqrt{\frac{4g}{\cos \phi}}$ ft. per sec.; g is the acceleration of gravity in ft./sec.². Find the smallest radius of the fountain for which no water will be thrown over the edge of the fountain.

Ans. $R = 8.485$ ft.

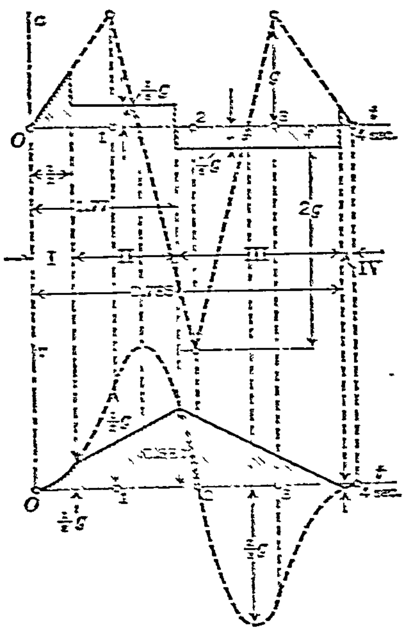
767. The bob of a clock pendulum carries a small weight, the position of which can be changed to give fine adjustments to the natural frequency of the pendulum. W is the weight of the pendulum bob, h is the distance between the point of suspension and the center of gravity of the bob, and l is the length of the equivalent simple pendulum. The weight of the adjustable weight is w and x is its distance from the point of suspension. Find the change Δl in the length of the equivalent pendulum for different values of w and x. Find the value of $x = x_1$ for which a given change Δl will be effected by a minimum additional weight w.

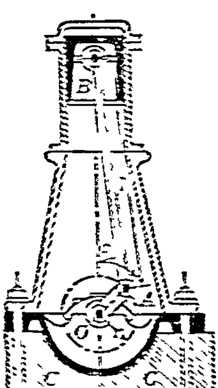
$$Ans. \quad \Delta l = \frac{wx(x - l)}{Wh + wx}; \; x_1 = \tfrac{1}{2}(l + \Delta l).$$

768. A horizontal steel trough conveying coal oscillates back and forth; the period of motion is 4 sec. At the beginning of each period the velocities of the trough and the coal are both zero. The acceleration of the trough changes every quarter period, as shown in the acceleration-time diagram. The coefficient of friction between steel and coal at rest is $k_1 = 0.5$ and in motion it is $k_2 = 0.2$. Draw the diagram of acceleration for

the coal, and the diagrams of velocity for the trough and for the coal. Find the distance the coal advances during each period.

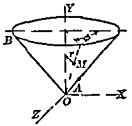
Ans. 23 ft.





769. The cylinder, crank case, and bearings of a vertical gas engine weigh 20,000 lbs. The piston weighs 1932 lbs. Its center of gravity is at B on the center line BO. The stroke of the piston is 24 in. and the crank revolves at 300 r.p.m. The ratio of the crank radius r to the connecting-rod length l is $r/l = 1/6$. The engine is held on its foundation by bolts C which are unstressed when the engine is not running. Find the maximum force N acting on the foundation and the total tension T in the bolts. Neglect the effects of the weight of the crank and the connecting rod. Neglect all terms which have factors of r/l raised to the second or higher powers. Find the weight Q of foundation C such that the inertia forces will not produce oscillations with an amplitude greater than 0.010 in.

Ans. $N = 82{,}000$ lbs.; $T = 38{,}000$ lbs.; $Q = 4{,}614{,}000$ lbs.

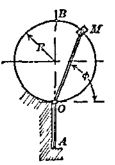


770 A particle M of weight w lbs moves on the smooth surface of a right circular cone which has an angle of 90° at its vertex A force of repulsion F, which is proportional to the distance OM, acts on the particle

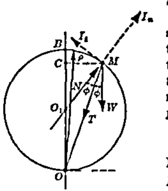
$$F = \frac{w}{g} OM \text{ lbs}$$

At time $t = 0$, M is at A, where $OA = 2$ in, and it has velocity $v_0 = 2$ in per sec directed parallel to the base of the cone Find the motion of M

Ans $r^2 = e^{2t} + e^{-2t}$, $\phi = \sqrt{2}(\tan^{-1} e^{2t} - \pi/4)$



771 A smooth ring M weighing 1 oz slides without friction on a circular wire loop of radius R The plane of the loop is vertical The ring is attached to an elastic cord MOA which passes through an immovable smooth ring at O and is fixed at A The unstretched length of MOA is OA The spring characteristic of the cord is k oz per inch elongation At time $t = 0$, M is at B in unstable equilibrium A slight displacement to the right starts it sliding down the wire Find the force exerted by the ring on the wire at any point



Solution

The inertia forces and the applied forces acting on the ring are in equilibrium Project all these forces on the direction of radius O_1M The tangential inertia force I_t has no component in that direction and the normal inertia force I_n is in equilibrium with the reaction N and the projections of cord tension T and weight W on O_1M

$$I_n + N - W\cos 2\phi - T\cos\phi = 0,$$

$$N = T\cos\phi + W\cos 2\phi - I_n$$

But $T = k \quad OM = 2kR\cos\phi$,

$I_n = (W/g)(v^2/R)$ The velocity v of the ring is found from the work done by W and T on the ring (§ 118) We have

$$\frac{1}{2}\frac{W}{g}v^2 = W \quad BC + \int_{OM=2R\cos\phi}^{OB=2R} k\rho\, d\rho = 2R(W + kR)\sin^2\phi$$

and

$$I_n = \frac{Wr^2}{gR} = 4(W + kR)\sin^2\phi.$$

Substituting, we find $N = -[kR + 2W - 3(W + kR)\cos 2\phi]$. The force of the ring on the wire, with $W = 1$ oz., is

$$F = kR + 2 - 3(1 + kR)\cos 2\phi \text{ oz.}$$

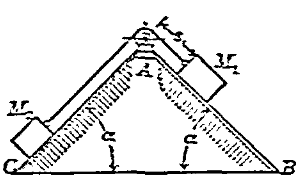
772. A string OM of length l, fixed at O, carries a body M of weight p on its free end. At time $t = 0$, the pendulum is pulled to one side until OM is at an angle α to the vertical, and then it is released. During the motion the string hits a wire O_1 which is stretched normal to the plane of the motion of OM. The position of O_1 in the plane is given by the distance $OO_1 = h$ and the angle β. Find the minimum value of α at which the string will wind around O_1 after it hits. Find the change $T_2 - T_1$ in the string tension at the instant of hitting the wire. Neglect the wire dimensions.

$$Ans. \quad \alpha = \cos^{-1}\left\{\frac{h}{l}(3/2 + \cos\beta) - 3/2\right\};$$

$$T_2 - T_1 = \frac{2ph}{l}(\cos\beta + 3/2).$$

773. Part of a thread of total length L lies on a smooth horizontal table. The other part of length l hangs over the edge of the table. When released, the thread slides off the table under the action of the weight of the hanging part. The initial velocity of the thread is zero. Find the time T taken by the thread to slide off the table.

$$Ans. \quad T = \sqrt{\frac{L}{g}}\log\frac{L + \sqrt{L^2 - l^2}}{l}.$$



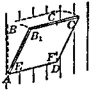
774. Two carriages M_1 and M_2, weighing p_1 and p_2, roll on rails AB and AC which are both inclined at an angle α to the horizontal. The carriages are tied together by means of a cable of length l. The cable passes over a pulley at A. At time $t = 0$, M_1 is at a distance a from A and its velocity is zero. Assuming the weight of the cable to be negligible and that $p_1 > p_2$, find the motion s_1 of M_1. Assuming that the weight of the

cable is p per unit length and that $p_1 > p_2 + pl$, find the motion s_2 of M_1

$$Ans \quad s_1 = a + \frac{p_1 - p_2}{p_1 + p_2} g \frac{t^2}{2} \sin\alpha,$$

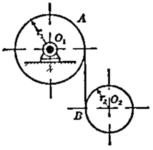
$$s_2 = a + \frac{p_1 - p_2 - pl + 2pa}{4p}(e^{nt} - e^{-nt})^2,$$

$$\text{where} \quad n = \sqrt{\frac{1}{2}\frac{pg \sin\alpha}{p_1 + p_2 + pl}}.$$



775 A thin rectangular board $ABCD$ of weight Q and height $AB = 2$ ft stands on two short headless nails E and F and leans against a wall $AE = FD$ The board starts to fall with a negligible velocity, rotating about AD What angle α will the board make with the wall at the instant it leaves the nails?

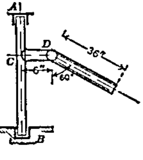
Ans $\alpha = 48°$



776 Two solid circular cylinders A and B, of weights p_1 and p_2 and radii r_1 and r_2, have two strings wound around them The cylinder A can rotate about a fixed axis The cylinder B starts from rest and falls under the action of gravity Find the angular velocities ω_1 and ω_2 of the two cylinders, the distance S traversed by the center O_2 and the tension T in each string as a function of time Assume that the strings do not become completely unwound

$$Ans \quad \omega_1 = \frac{2gp_2}{r_1(3p_1 + 2p_2)}t, \ \omega_2 = \frac{2gp_1}{r_2(3p_1 + 2p_2)}t,$$

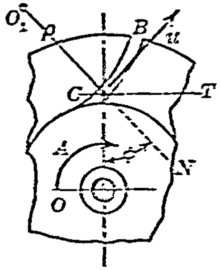
$$S = \frac{g(p_1 + p_2)}{3p_1 + 2p_2}t^2, \quad T = \frac{p_1 p_2}{2(3p_1 + 2p_2)}.$$



777. In the diagram shown AB is a vertical shaft with a rigid arm CD A uniform slender bar, weighing 16 1 lbs, is connected to the rigid arm CD by a smooth pin at D The entire system rotates at a constant speed about the vertical axis Calculate the angular velocity of the system and determine all

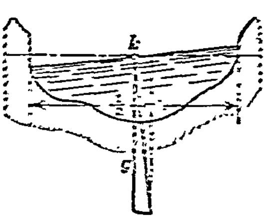
the forces acting on the hinged bar when the angle between the bar and the vertical is 60°.

Ans. $\omega = 5.56$ rad. per sec.; $D_x = -27.8$ lbs.; $D_y = +16.1$ lbs.



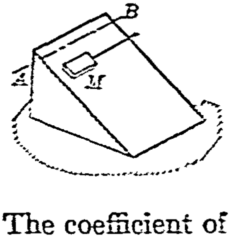
778. Solve Problem 442 for the case of curved vanes. The radius of curvature of the water channel at C is ρ and the angle between the radius of curvature and OC is ϕ. Find the projection a_n of the acceleration on the direction of the velocity u. Find the moment M about the center O exerted by the force of the particle C against the smooth vane. The weight of the particle is w. *Ans.* $M = \frac{w}{g} r \left(\frac{u^2}{\rho} - 2u\omega \right) \sin \phi.$

779. An experimental railroad track extends North and South. An electric car weighing 200,000 lbs. runs north at a speed of 126 mi. per hr. Find the Coriolis acceleration a_{cor} and the corresponding force on the rails when the locomotive is at a latitude of 45° N. *Ans.* 118 lbs.



780. A river half a mile wide flows northward with a velocity of 3 mi. per hr. Find the Coriolis acceleration at the latitude 60°N. At which bank will the water level be higher? How much higher?
Ans. The level will be higher at the right bank by 0.045 ft.

781. A locomotive weighing 120,000 lbs. runs at a speed of 60 ft. per sec. along a track which extends East and West at a latitude of 30° North. Find the Coriolis acceleration a_{cor} of the locomotive and the corresponding additional force on the rail. *Ans.* 17.1 lbs.



782. A body M weighing 3 lbs. moves on a rough plane inclined at an angle $\alpha = \tan^{-1} \frac{1}{30}$. It has a constant pull exerted on it by a string parallel to AB. After a certain time the motion becomes uniform and rectilinear; the component of velocity parallel to AB is 12 in./sec. The coefficient of friction is 0.1. Find the component of velocity v normal to AB and the tension T in the string.

Ans. $v = 4.24$ in./sec.; $T = 0.28$ lb.

APPENDIX

TABLE OF UNITS

Length:

1 mile = 5280 ft. = 1.609 km.
1 foot = 12 in. = 30.48 cm.
1 inch = 2.540 cm. = 25.40 mm.

Velocity:

1 mile per hour = 88 ft. per minute = 1.467 ft. per sec.
1 foot per sec. = 0.3048 meters per sec.
1 cm. per sec. = 0.0328 ft. per sec.

Acceleration:

1 mile per hour per sec. = 1.467 ft. per sec. per sec. = 1.467 ft./sec.^{2}
Acceleration of gravity g = 32.2 ft./sec.^{2} = 386 in./sec.^{2}
= 9.80 meters/sec.^{2}
= 980 cm./sec.^{2} (approximately).

Force:

1 pound = 16 ounces = 0.454 kg.
1 short ton = 2000 lbs.
1 long ton = 2240 lbs.
1 metric ton = 1000 kgs. = 2205 lbs.

Moment:

1 pound-foot = 12 pound-inches = 0.1383 kg.-m. = 13.83 kg.-cm.

Work and Energy:

1 foot-pound = 12 inch-pounds = 0.1383 kg.-m. = 13.83 kg.-cm.
1 BTU = 777.5 foot-pounds.
1 kilogram calorie = 426.6 kg.-m.

Power:

1 horsepower = 550 foot-pounds per second
= 33,000 foot-pounds per minute.
1 metric horsepower = 0.9863 horsepower = 75 kg.-m. per second.

Systems of Units

System	Engineering Systems			Absolute Systems		
	Dimensions	British or foot-pound second	Metric or meter kilogram second	Dimensions	Metric or centimeter-gram-second	British or pound mass-foot-second
Length	L	1 foot	1 meter	L	1 centimeter	1 foot
Time	T	1 second	1 second	T	1 second	1 second
Force	F	1 pound	1 kilogram	MLT^{-2}	1 dyne = 1 gr cm sec $^{-2}$	1 poundal = 1 lb ft sec $^{-2}$
Mass	$FL^{-1}T^2$	1 lb ft $^{-1}$ sec ^{2} = 1 slug	1 kg m $^{-1}$ sec ^{2}	M	1 gram	1 pound mass
		One slug weighs 32 2 lbs	Unit mass weighs 9 80 kilograms		1 gram weighs 980 dynes	1 pound mass weighs 32 2 poundals

(The relation between the engineering and absolute systems is
1 pound mass weighs 1 pound at the "standard location," where g = 32 2 ft /sec ^{2},
1000 grams weigh 1 kilogram at the "standard location," where g = 980 cm /sec ^{2})